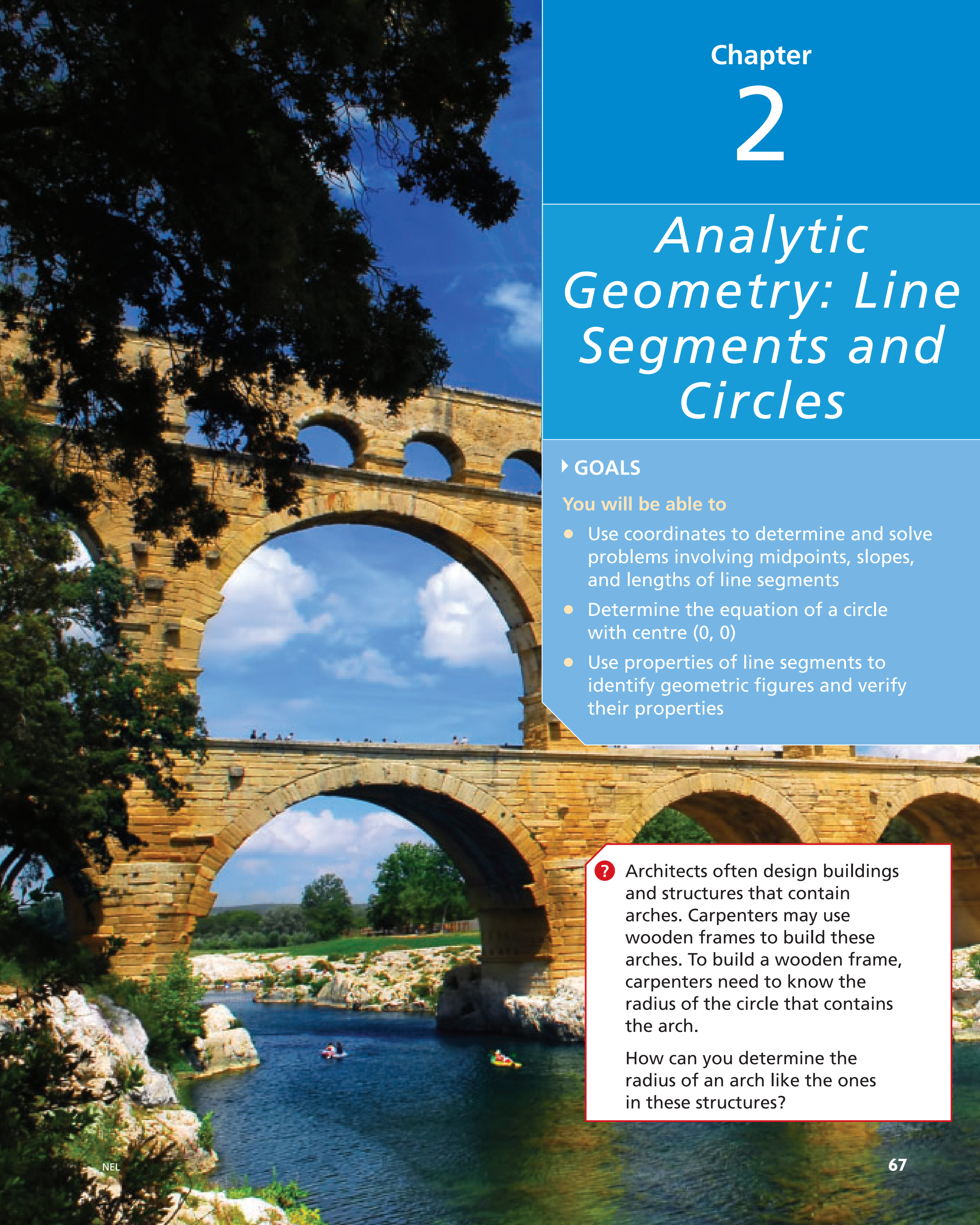


Chapter

2

Analytic Geometry: Line Segments and Circles

GOALS

You will be able to

- Use coordinates to determine and solve problems involving midpoints, slopes, and lengths of line segments
- Determine the equation of a circle with centre (0, 0)
- Use properties of line segments to identify geometric figures and verify their properties

? Architects often design buildings and structures that contain arches. Carpenters may use wooden frames to build these arches. To build a wooden frame, carpenters need to know the radius of the circle that contains the arch.

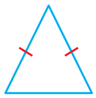
How can you determine the radius of an arch like the ones in these structures?

2 Getting Started

WORDS YOU NEED to Know

1. Match each word with the diagram that best represents it.

a) diagonal
b) scalene triangle
c) perpendicular bisector
d) median of a triangle
e) midsegment of a triangle
f) Cartesian coordinate system
g) isosceles triangle
h) equilateral triangle

i)


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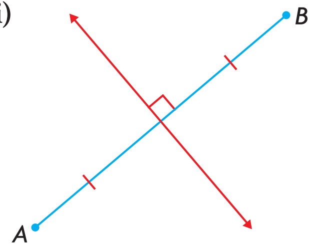

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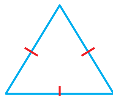

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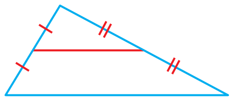

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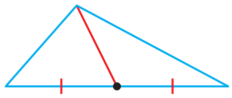

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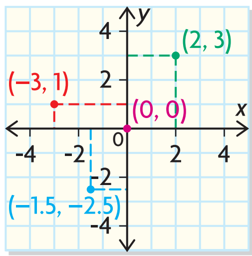

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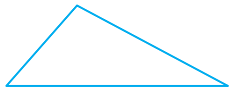

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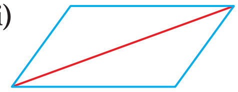
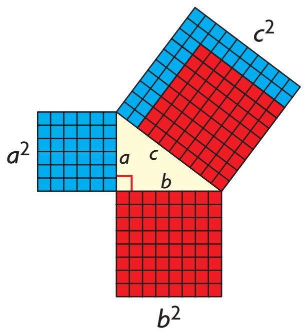


SKILLS AND CONCEPTS You Need

The Pythagorean Theorem

Study Aid

- For more help and practice, see Appendix A-4.



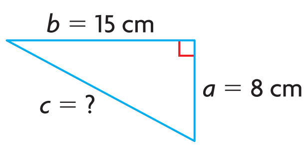
The **hypotenuse** is the longest side in a right triangle. If c represents the hypotenuse, and a and b represent the other two sides, $a^2 + b^2 = c^2$.

EXAMPLE

A right triangle has two perpendicular sides that measure 15 cm and 8 cm. Calculate the length of the hypotenuse.

Solution

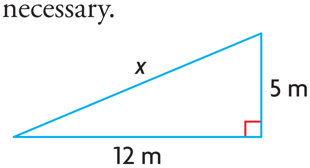
Draw a right triangle with $a = 8$, $b = 15$, and c as the hypotenuse.



$$\begin{aligned} a^2 + b^2 &= c^2 \\ 8^2 + 15^2 &= c^2 \\ 289 &= c^2 \\ \sqrt{289} &= c \\ 17 &= c \end{aligned}$$

The hypotenuse is 17 cm long.

2. Calculate each indicated side length. Round to the nearest tenth, if necessary.

a)


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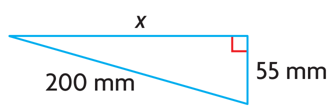
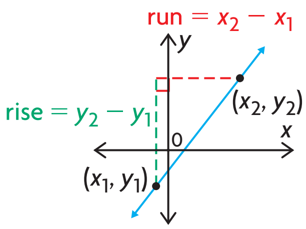
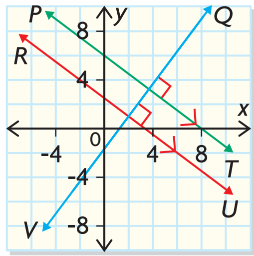


The Slope and the Equation of a Line





The slope of a line is the rise divided by the run:

$$m = \frac{\text{rise}}{\text{run}}$$

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

Parallel lines have the same slope:
$m_{PT} = m_{RU}$

Perpendicular lines have slopes that are negative reciprocals:

$$m_{QV} = -\frac{1}{m_{PT}};\ m_{QV} = -\frac{1}{m_{RU}}$$

The equation of a line may be written in

- slope y-intercept form: $y = mx + b$, where m is the slope and b is the y-intercept
- standard form: $Ax + By + C = 0$

EXAMPLE

a) Determine the equation of the line through $A(-1, 7)$ and $B(2, 6)$.
b) Show that line AB is perpendicular to the line $y = 3x - 2$.

Solution

a) ① Determine the slope of AB.

$$m_{AB} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{6 - 7}{2 - (-1)} = -\frac{1}{3}$$

② Determine the y-intercept of AB.

The equation is $y = -\frac{1}{3}x + b$; $(x, y) = (2, 6)$.

$$6 = -\frac{1}{3}(2) + b$$

$$6 + \frac{2}{3} = b$$

$$\frac{20}{3} = b$$

③ Write the equation.

The equation is $y = -\frac{1}{3}x + \frac{20}{3}$.

b) The line $y = 3x - 2$ has slope 3.

The negative reciprocal of 3 is $-\frac{1}{3}$, the slope of line AB. So, line AB, defined by $y = -\frac{1}{3}x + \frac{20}{3}$, is perpendicular to the line $y = 3x - 2$.

3. Determine the equation of the line that
a) passes through points $(-5, 3)$ and $(7, 7)$
b) is perpendicular to $y = \frac{1}{4}x + 7$ and passes through $(-1, -2)$
c) is parallel to $y = -5x + 6$ and passes through $(4, -3)$

Study **Aid**

- For help, see the Review of Essential Skills and Knowledge Appendix.

Question	Appendix
4	A-2, A-8
5a) to c)	A-9

PRACTICE

4. Simplify each expression.

a) $\frac{1}{2}(-6) + \frac{3}{2}$

b) $\frac{3}{8} - \frac{3}{7}$

c) $\frac{2}{3}x + 11x$

d) $\frac{3}{4}y - \frac{3}{8}y$

e) $\left(\frac{2}{3}\right)\left(\frac{3}{5}\right) + \frac{3}{4}$

f) $(-1.5)(0.625) + (4)(-0.125)$

5. Solve.

a) $3(7 - 4x) - \frac{4}{3}(2x + 1) = 49$

b) $\frac{1}{4}(x + 3) + \frac{1}{3}(x - 2) = -\frac{1}{2}$

c) $\frac{x + 4}{4} - \frac{x - 2}{3} = 1$

d) $x^2 = 36$

e) $x^2 + 16 = 25$

f) $225 + x^2 = 289$

6. Determine the **point of intersection** for each pair of lines.

a) $y = 2x + 5$
$y = 3x + 4$

b) $4x + 2y = 7$
$6x - 4y = 0$

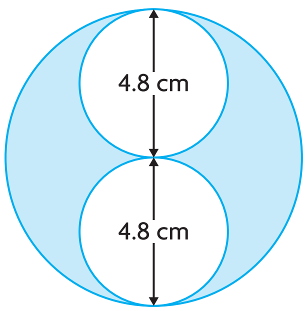
7. Determine the **mean** for each set of numbers.

a) $7, -11, 23, 5$

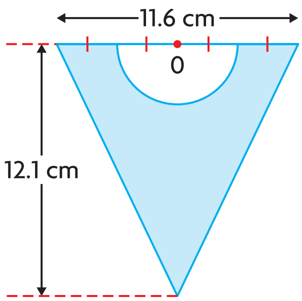
b) $-\frac{1}{6}, \frac{2}{3}$

c) $-1.4, 3.6, -0.1$

8. a) Calculate the area of the shaded region. Round to one decimal place.



b) Calculate the area and perimeter of the shaded region. Round to one decimal place.



9. Draw a Venn diagram to show relationships for the following figures.

quadrilateral	trapezoid
square	parallelogram
rectangle	rhombus

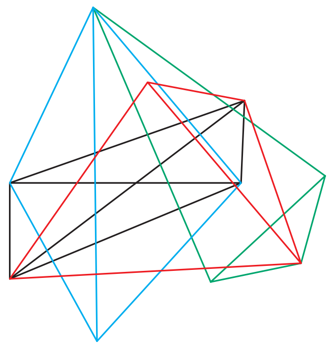
APPLYING *What You Know*

Diagonal Patterns

Jamie was creating a picture for her art portfolio using combinations of intersecting **line segments**. She noticed that whenever she joined the endpoints of a pair of these line segments, she created a **quadrilateral**.

YOU WILL NEED

- grid paper, ruler, and protractor, or dynamic geometry software



? How can you predict the type of quadrilateral by using the properties of its diagonals?

A. On a grid, plot four points and join them to create a square.

B. Verify that your figure is a square by measuring its sides and **interior angles**.

C. Draw the diagonals in your figure, and measure
- **i)** their lengths
- **ii)** the angles formed at their point of intersection

D. What do you notice about the diagonals you drew for part C?

E. Repeat parts A to D for several more squares to determine whether your observations are the same.

F. Copy the table, and record your observations for parts A to E.

Type of Quadrilateral	Side Relationships	Interior Angle Relationships	Diagonal Relationships	Relationship of Angles Formed by Intersecting Diagonals	Diagram
square					
rectangle					
parallelogram					
rhombus					
isosceles trapezoid					
kite					

Tech ***Support***

For help using dynamic geometry software to plot points, construct lines, and measure lengths and angles, see Appendix B-18, B-21, B-29, and B-26.

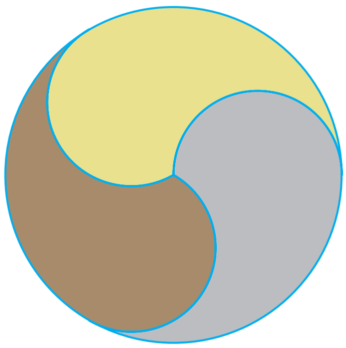
G. Repeat parts A to E for each of the other figures in the table. Record your observations.

H. Explain how you could use what you learned to help you distinguish between the figures in each pair.
- **i)** a square and a rhombus
- **ii)** a square and a rectangle
- **iii)** a rhombus and a parallelogram
- **iv)** a rhombus and a kite

2.1 Midpoint of a Line Segment

YOU WILL NEED

- grid paper, ruler, and compass, or dynamic geometry software

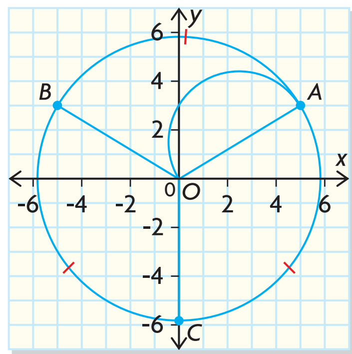


GOAL

Develop and use the formula for the midpoint of a line segment.

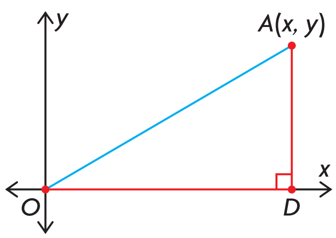
INVESTIGATE the Math

Ken's circular patio design for a client is shown at the left. He is planning the layout on a grid. He starts by drawing a circle that is centred at the origin. Then he marks points *A*, *B*, and *C* on the **circumference** of the circle to divide it into thirds. He joins these points to point *O*, at the centre of the circle. He needs to draw semicircles on the three **radii**: *OA*, *OB*, and *OC*.



? How can Ken determine the coordinates of the centre of the semicircle he needs to draw on radius *OA*?

A. Construct a line segment like *OA* on a coordinate grid, with *O* at (0, 0) and *A* at a grid point. Name the coordinates of *A*(*x*, *y*).



B. Draw right triangle *OAD*, with side *OD* on the *x*-axis and side *OA* as the hypotenuse.

C. Draw a vertical line from *E*, the **midpoint** of *OD*, to *M*, the midpoint of *OA*. Explain why △*OME* is similar to △*OAD*. Explain how the sides of the triangles are related. Estimate the coordinates of *M*.

D. Record the coordinates of point *M*. Explain why this is the centre of the semicircle that Ken needs to draw.

Reflecting

E. Why does it make sense that the coordinates of point *M* are the means of the coordinates of points *O* and *A*?

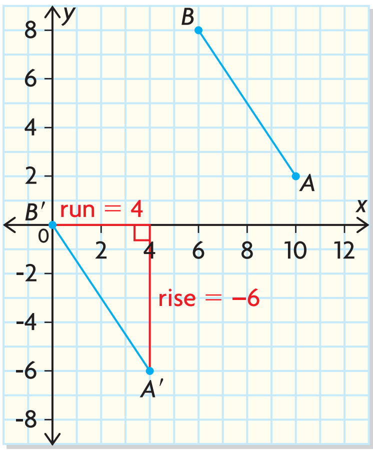
F. Suppose that point *O* had not been at (0, 0) but at another point instead. If (x_1, y_1) and (x_2, y_2) are endpoints of a line segment, what formula can you write to represent the coordinates of the midpoint? Why does your formula make sense?

APPLY the Math

EXAMPLE 1 Reasoning about the midpoint formula when one endpoint is not the origin

Determine the midpoint of a line segment with endpoints $A(10, 2)$ and $B(6, 8)$.

Robin's Solution: Using translations



I drew *AB* by plotting points *A* and *B* on a grid and joining them.

To make it easier to calculate the midpoint of *AB*, I decided to translate *AB* so that one endpoint would be at the origin. I moved point *B* to the origin by translating it 6 units left and 8 units down. I did the same to point *A* to get (4, −6) for *A′*.

I could see that the run of *A′B′* was 4 and the rise was −6.

$B'(6 - 6, 8 - 8) = B'(0, 0)$
$A'(10 - 6, 2 - 8) = A'(4, -6)$

x-coordinate of midpoint M'

$= 0 + \frac{4}{2}$

$= 2$

I determined the *x*-coordinate of the midpoint of *A′B′* by adding half the run to the *x*-coordinate of *B′*.

y-coordinate of midpoint M'

$= 0 + \frac{-6}{2}$

$= -3$

I determined the *y*-coordinate of the midpoint of *A′B′* by adding half the rise to the *y*-coordinate of *B′*.

The midpoint of line segment $A'B'$ is $(2, -3)$.

$M_{AB} = M(2 + 6, -3 + 8)$
$M_{AB} = (8, 5)$

To determine the coordinates of *M*, the midpoint of *AB*, I had to undo my translation. I added 6 to the *x*-coordinate of the midpoint and 8 to the *y*-coordinate.

The midpoint of line segment *AB* is (8, 5).

Sarah's Solution: Calculating using a formula

$$(x, y) = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$

I decided to use the midpoint formula.

$$x_1 = 10, y_1 = 2$$
$$x_2 = 6, y_2 = 8$$

I chose point $A(10, 2)$ to be (x_1, y_1) and point $B(6, 8)$ to be (x_2, y_2).

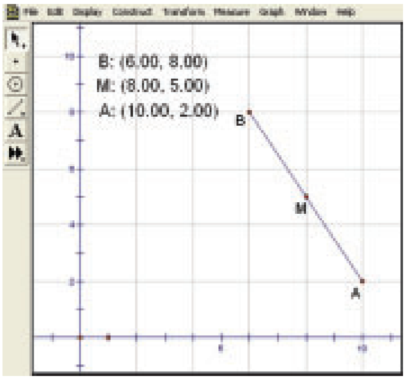
$$(x, y) = \left(\frac{10 + 6}{2}, \frac{2 + 8}{2}\right)$$
$$= \left(\frac{16}{2}, \frac{10}{2}\right)$$
$$= (8, 5)$$

I substituted these values into the midpoint formula.

The midpoint of line segment AB is (8, 5).

Tech Support

For help constructing and labelling a line segment, displaying coordinates, and constructing the midpoint using dynamic geometry software, see Appendix B-21, B-22, B-20, and B-30.

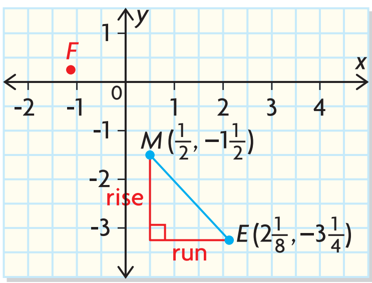


I verified my calculations by constructing AB using dynamic geometry software. Then I constructed the midpoint and measured the coordinates of all three points. My calculations were correct.

EXAMPLE 2 Reasoning to determine an endpoint

Line segment EF has an endpoint at $E\left(2\frac{1}{8}, -3\frac{1}{4}\right)$. Its midpoint is located at $M\left(\frac{1}{2}, -1\frac{1}{2}\right)$. Determine the coordinates of endpoint F.

Ali's Solution

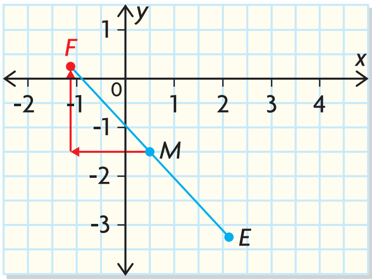


I reasoned that if I could calculate the run and rise between E and M, adding these values to the x- and y-coordinates of M would give me the x- and y-coordinates of F.

$$\text{Run} = x_2 - x_1 = 2\frac{1}{8} - \frac{1}{2} = \frac{17}{8} - \frac{4}{8} = \frac{13}{8} \text{ or } 1\frac{5}{8}$$

$$\text{Rise} = y_2 - y_1 = -3\frac{1}{4} - \left(-1\frac{1}{2}\right) = -\frac{13}{4} + \frac{6}{4} = -\frac{7}{4} \text{ or } -1\frac{3}{4}$$

I let $M = (x_1, y_1)$ and $E = (x_2, y_2)$. Then I calculated the rise and the run.



To get to F, I had to start at M and move $1\frac{5}{8}$ units left and $1\frac{3}{4}$ units up.

x-coordinate of F

$$= \frac{1}{2} - 1\frac{5}{8} = \frac{4}{8} - \frac{13}{8} = -\frac{9}{8} \text{ or } -1\frac{1}{8}$$

y-coordinate of F

$$= -1\frac{1}{2} + 1\frac{3}{4} = -\frac{6}{4} + \frac{7}{4} = \frac{1}{4}$$

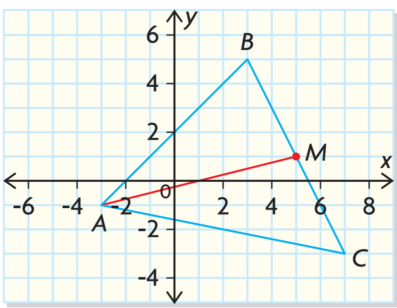
I subtracted $1\frac{5}{8}$ from the x-coordinate of M and added $1\frac{3}{4}$ to the y-coordinate.

The coordinates of F are $\left(-1\frac{1}{8}, \frac{1}{4}\right)$.

EXAMPLE 3 **Connecting** the midpoint to an equation of a line

A triangle has vertices at $A(-3, -1)$, $B(3, 5)$, and $C(7, -3)$. Determine an equation for the **median** from vertex A.

Graeme's Solution



I plotted A, B, and C and joined them to create a triangle.

I saw that the side opposite vertex A is BC, so I estimated the location of the midpoint of BC. I called this point M. Then I drew the median from vertex A by drawing a straight line from point A to M.

$$M = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$

$$M_{BC} = \left(\frac{3 + 7}{2}, \frac{5 + (-3)}{2}\right)$$

$$= (5, 1)$$

I used the midpoint formula to calculate the coordinates of *M*.

$$\text{Slope} = \frac{y_2 - y_1}{x_2 - x_1}$$

$$m_{AM} = \frac{1 - (-1)}{5 - (-3)}$$

$$= \frac{2}{8}$$

$$= \frac{1}{4}$$

To determine the equation of *AM*, I had to calculate its slope. I used the coordinates of *A* as (x_1, y_1) and the coordinates of *M* as (x_2, y_2) in the slope formula.

An equation for *AM* is

$$y = \frac{1}{4}x + b$$

I substituted the slope of *AM* for *m* in $y = mx + b$.

$$-1 = \frac{1}{4}(-3) + b$$

$$-1 + \frac{3}{4} = b$$

$$-\frac{1}{4} = b$$

Then I determined the value of *b* by substituting the coordinates of *A* into the equation and solving for *b*.

The equation of the median is $y = \frac{1}{4}x - \frac{1}{4}$.

EXAMPLE 4 Solving a problem using midpoints

A waste management company is planning to build a landfill in a rural area. To balance the impact on the two closest towns, the company wants the landfill to be the same distance from each town. On a coordinate map of the area, the towns are at $A(1, 8)$ and $B(5, 2)$. Describe all the possible locations for the landfill.

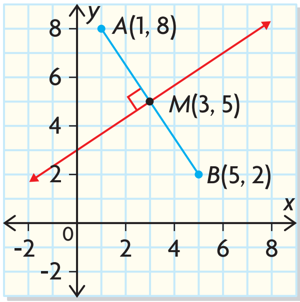
Wendy's Solution

$$M = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$

$$M_{AB} = \left(\frac{1 + 5}{2}, \frac{8 + 2}{2}\right)$$

$$= (3, 5)$$

I used the midpoint formula to determine the coordinates of the midpoint of *AB*.



I drew *AB* on a grid. I knew that the points equally far from *A* and *B* lie on the **perpendicular bisector** of *AB*, so I added this to my sketch.

$$(x_1, y_1) = A(1, 8)$$
$$(x_2, y_2) = B(5, 2)$$

$$\text{Slope of } AB = \frac{y_2 - y_1}{x_2 - x_1}$$
$$= \frac{2 - 8}{5 - 1}$$
$$= \frac{-6}{4}$$
$$= -\frac{3}{2}$$

I needed the slope of the perpendicular bisector so that I could write an equation for it. I used the slope formula to determine the slope of *AB*.

The slope of the perpendicular bisector is $\frac{2}{3}$.

Since the perpendicular bisector is perpendicular to *AB*, its slope is the negative reciprocal of the slope of *AB*.

An equation for the perpendicular bisector is

$$y = \frac{2}{3}x + b$$
$$5 = \frac{2}{3}(3) + b$$
$$5 = 2 + b$$
$$5 - 2 = b$$
$$3 = b$$

To determine the value of *b*, I substituted the coordinates of the midpoint of *AB* into the equation and solved for *b*. This worked because the midpoint is on the perpendicular bisector, even though points *A* and *B* aren't.

Therefore, $y = \frac{2}{3}x + 3$ is the equation of the perpendicular bisector. Possible locations for the landfill are determined by points that lie on the line with equation $y = \frac{2}{3}x + 3$.

Communication *Tip*

A perpendicular bisector is also called a right bisector.

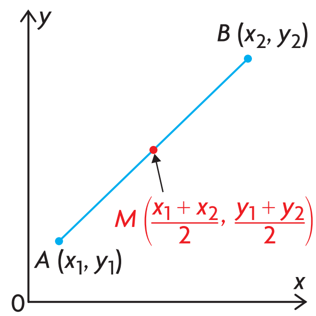
In Summary

Key Idea

- The coordinates of the midpoint of a line segment are the means of the coordinates of the endpoints.

Need to Know

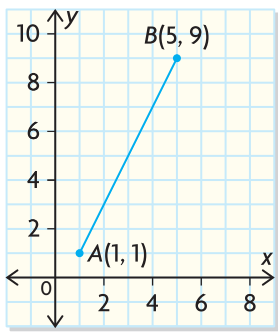
- The formula $(x, y) = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$ can be used to calculate the coordinates of a midpoint.
- The coordinates of a midpoint can be used to determine an equation for a median in a triangle or the perpendicular bisector of a line segment.



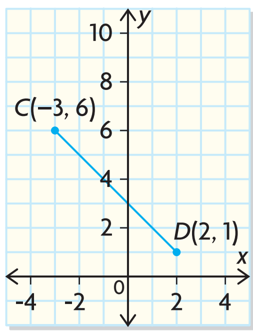
CHECK Your Understanding

1. Determine the coordinates of the midpoint of each line segment, using one endpoint and the rise and run. Verify the midpoint by measuring with a ruler.

a)

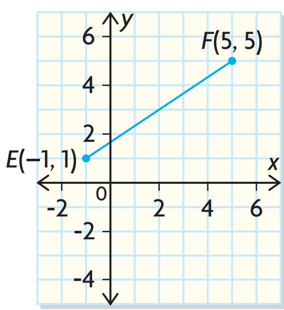


b)

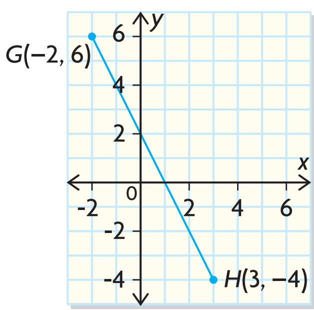


2. Determine the coordinates of the midpoint of each line segment.

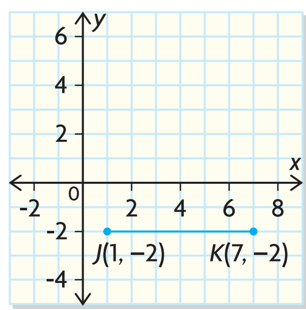
a)



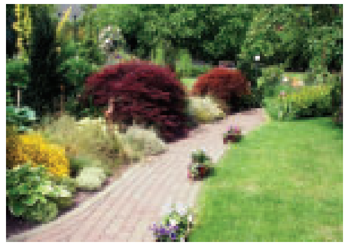
b)



c)



3. On the design plan for a landscaping project, a straight path runs from (11, 29) to (53, 9). A lamp is going to be placed halfway along the path.
   a) Draw a diagram that shows the path.
   b) Determine the coordinates of the lamp on your diagram.



PRACTISING

4. Determine the coordinates of the midpoint of the line segment with each pair of endpoints.
   a) $A(-1, 3)$ and $B(5, 7)$
   b) $J(-2, 3)$ and $K(3, 4)$
   c) $X(6, -2)$ and $Y(-2, -2)$
   d) $P(2, -4)$ and $I(-3, 5)$
   e) $U\left(\frac{1}{2}, -\frac{3}{2}\right)$ and $V\left(-\frac{5}{2}, -\frac{1}{2}\right)$
   f) $G(1.5, -2.5)$ and $H(-1, 4)$

5. The endpoints of the diameter of a circle are $A(-1, 1)$ and $B(2.5, -3)$. Determine the coordinates of the centre of the circle.

6. $P(-3, -1)$ is one endpoint of PQ. $M(1, 1)$ is the midpoint of PQ. Determine the coordinates of endpoint Q. Explain your solution.

7. A triangle has vertices at $A(2, -2)$, $B(-4, -4)$, and $C(0, 4)$.
   K **a)** Draw the triangle, and determine the coordinates of the midpoints of its sides.
   b) Draw the median from vertex A, and determine its equation.

8. A radius of a circle has endpoints $O(-1, 3)$ and $R(2, 2)$. Determine the endpoints of the diameter of this circle. Describe any assumptions you make.

9. A quadrilateral has vertices at $P(1, 3)$, $Q(6, 5)$, $R(8, 0)$, and $S(3, -2)$. Determine whether the diagonals have the same midpoint.

10. C Mayda is sketching her design for a rectangular garden. By mistake, she has erased the coordinates of one of the corners of the garden. As a result, she knows only the coordinates of three of the rectangle's vertices. Explain how Mayda can use midpoints to determine the unknown coordinates of the fourth vertex of the rectangle.

11. A triangle has vertices at $P(7, 7)$, $Q(-3, -5)$, and $R(5, -3)$.
   A **a)** Determine the coordinates of the midpoints of the three sides of $\triangle PQR$.
   b) Calculate the slopes of the **midsegments** of $\triangle PQR$.
   c) Calculate the slopes of the three sides of $\triangle PQR$.
   d) Compare your answers for parts b) and c). What do you notice?

12. Determine the equations of the medians of a triangle with vertices at $K(2, 5)$, $L(4, -1)$, and $M(-2, -5)$.

13. Determine an equation for the perpendicular bisector of a line segment with each pair of endpoints.

a) $C(-2, 0)$ and $D(4, -4)$

b) $A(4, 6)$ and $B(12, -4)$

c) $L(-2, -4)$ and $M(8, 4)$

d) $Q(-5, 6)$ and $R(1, -2)$

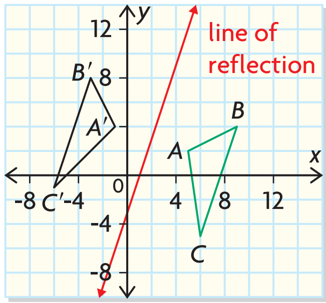
Health *Connection*

Vegetables, a source of vitamins and minerals, lower blood pressure, reduce the risk of stroke and heart disease, and decrease the chance of certain types of cancer.

14. A committee is choosing a site for a county fair. The site needs to be located the same distance from the two main towns in the county. On a map, these towns have coordinates (3, 10) and (13, 4). Determine an equation for the line that shows all the possible sites for the fair.

15. **T** A triangle has vertices at $D(8, 7)$, $E(-4, 1)$, and $F(8, 1)$. Determine the coordinates of the point of intersection of the medians.

16. In the diagram, $\triangle A'B'C'$ is a reflection of $\triangle ABC$. The coordinates of all vertices are integers.



a) Determine the equation of the line of reflection.

b) Determine the equations of the perpendicular bisectors of AA', BB', and CC'.

c) Compare your answers for parts a) and b). What do you notice?

17. A quadrilateral has vertices at $W(-7, -4)$, $X(-3, 1)$, $Y(4, 2)$, and $Z(-2, -7)$. Two lines are drawn to join the midpoints of the non-adjacent sides in the quadrilateral. Determine the coordinates of the point of intersection of these lines.

18. Describe two different strategies you can use to determine the coordinates of the midpoint of a line segment using its endpoints. Explain how these strategies are similar and how they are different.

Extending

19. A point is one-third of the way from point $A(1, 7)$ to point $B(10, 4)$. Determine the coordinates of this point. Explain the strategy you used.

20. A triangle has vertices at $S(6, 6)$, $T(-6, 12)$, and $U(0, -12)$. SM is the median from vertex S.

a) Determine the coordinates of the point that is two-thirds of the way from S to M that lies on SM.

b) Repeat part a) for the other two medians, TN and UR.

c) Show that the three medians intersect at a common point. What do you notice about this point?

d) Do you think the relationship you noticed is true for all triangles? Explain.

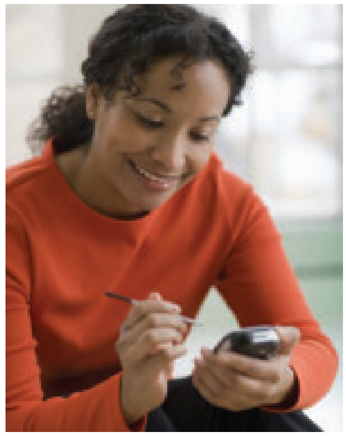
2.2 Length of a Line Segment

GOAL

Determine the length of a line segment.

YOU WILL NEED

- grid paper
- ruler



INVESTIGATE the Math

Some computers can translate a handwritten entry into text by calculating the lengths of small line segments within the entry and comparing these lengths to stored information about the lengths of pieces of letters.

? How can you use the coordinates of the endpoints of a line segment to determine its length?

A. Plot two points, *A* and *B*, on a grid so that line segment *AB* is neither horizontal nor vertical. Join *A* and *B*. Then construct a right triangle that has *AB* as its hypotenuse.

B. Write the coordinates of the vertex of the right angle. How are these coordinates related to the coordinates of the endpoints of the right angle?

C. Determine the lengths of the horizontal and vertical sides of the right triangle. How are these lengths related to the coordinates of *A* and *B*?

D. Calculate the length of *AB*.

E. Repeat parts A to D for line segment *PQ*, with endpoints $P(x_1, y_1)$ and $Q(x_2, y_2)$.

Reflecting

F. Does it matter which point is (x_1, y_1), and which is (x_2, y_2), when calculating the length of a line segment? Explain.

G. Describe how to use each of the four coordinates of points *P* and *Q* to determine the length of *PQ*.

H. Why do you think the equation for calculating the length of a line segment is sometimes called the distance formula?

APPLY the Math

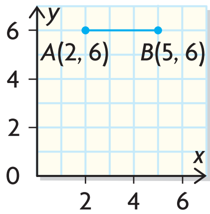
EXAMPLE 1 **Selecting a strategy** to calculate the length of a line segment

Determine the length of the line segment with each pair of endpoints.

a) $A(2, 6)$ and $B(5, 6)$ **b)** $G(-7, 8)$ and $H(-7, -5)$ **c)** $P(-4, 7)$ and $Q(3, 1)$

Niranjan's Solution

a)



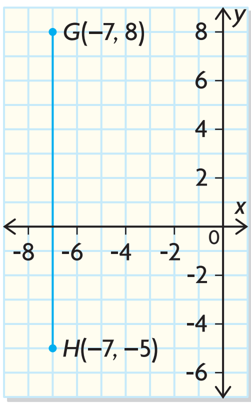
I noticed that *A* and *B* have the same *y*-coordinate, so I knew that *AB* was horizontal. I made a sketch to check.

$$AB = 5 - 2$$
$$= 3$$

I calculated the difference in the *x*-coordinates to determine the length of *AB*.

The length of *AB* is 3 units.

b)



I noticed that *G* and *H* have the same *x*-coordinate, so I knew that *GH* was vertical. I made a sketch to check.

$$GH = 8 - (-5)$$
$$= 13$$

I calculated the difference in the *y*-coordinates to determine the length of *GH.*

The length of *GH* is 13 units.

c) $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

I noticed that the *x*- and *y*-coordinates of the endpoints are different numbers. I knew the line segment couldn't be horizontal or vertical. So, I used the distance formula.

$$PQ = \sqrt{[3 - (-4)]^2 + (1 - 7)^2}$$
$$= \sqrt{7^2 + (-6)^2}$$
$$= \sqrt{49 + 36}$$
$$= \sqrt{85}$$
$$\doteq 9.2$$

I chose $P(-4, 7)$ to be (x_1, y_1) and $Q(3, 1)$ to be (x_2, y_2). I substituted these values into the distance formula.

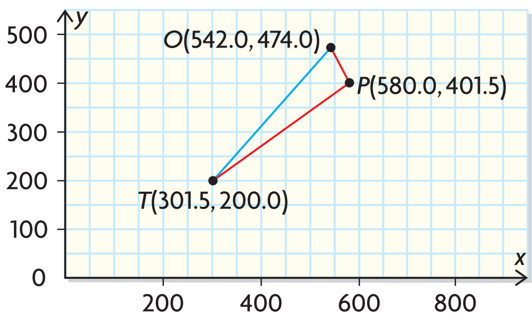
The length of *PQ* is approximately 9.2 units.

I rounded my answer to the nearest tenth of a unit.

EXAMPLE 2 Representing distances on a coordinate grid

Winston takes different routes to drive from his home in Toronto to Carleton University in Ottawa. Sometimes he takes Highway 401 to Prescott, and then Highway 416 to Ottawa. Other times he drives directly from Toronto to Ottawa along Highway 7. On a map of southeastern Ontario, with the origin at Windsor and the coordinates in kilometres, Toronto is at $T(301.5, 200.0)$, Prescott is at $P(580.0, 401.5)$, and Ottawa is at $O(542.0, 474.0)$. Approximately how far does Winston drive using each route?

Marla's Solution



I made a sketch on grid paper. I saw that I had to calculate the lengths of *TO*, *TP*, and *PO*, and then compare *TO* with *TP* + *PO*.

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$
$$TO = \sqrt{(542.0 - 301.5)^2 + (474.0 - 200.0)^2}$$
$$= \sqrt{240.5^2 + 274.0^2}$$
$$= \sqrt{57\,840.25 + 75\,076.00}$$
$$= \sqrt{132\,916.25}$$
$$\doteq 365$$

I used the distance formula to determine the length of each line segment.

I rounded my first answer to the nearest kilometre to determine the distance that Winston drives using the direct route.

$$TP = \sqrt{(580.0 - 301.5)^2 + (401.5 - 200.0)^2}$$
$$= \sqrt{278.5^2 + 201.5^2}$$
$$= \sqrt{77\,562.25 + 40\,602.25}$$
$$= \sqrt{118\,164.5}$$
$$\doteq 343.8$$

I knew that I was going to add the lengths *TP* and *PO*, so I rounded to the nearest tenth of a kilometre because I planned to use these distances in another calculation.

$$PO = \sqrt{(542.0 - 580.0)^2 + (474.0 - 401.5)^2}$$
$$= \sqrt{(-38.0)^2 + 72.5^2}$$
$$= \sqrt{1444.00 + 5256.25}$$
$$= \sqrt{6700.25}$$
$$\doteq 81.9$$

$TP + PO = 343.8 + 81.9$
$\doteq 426$

I added lengths *TP* and *PO* to determine the distance that Winston drives using the indirect route. I rounded the distance to the nearest kilometre.

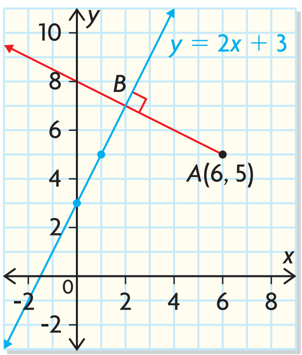
The route from Toronto directly to Ottawa is approximately 365 km. The route through Prescott is approximately 426 km.

These distances are estimates because they don't take into account turns in the road. Even though the route along Highways 401 and 416 is longer, it might be faster since Winston can travel at a greater speed on multi-lane highways.

EXAMPLE 3 Reasoning to determine the distance between a point and a line

Calculate the distance between point $A(6, 5)$ and the line $y = 2x + 3$.

Kerry's Solution



I graphed the line by plotting the *y*-intercept at (0, 3). Then I used the slope to determine a second point on the line. I drew a straight line between these points. I also plotted point *A*.

I reasoned that the distance I needed to calculate was the shortest distance between point *A* and the line. I figured that this distance would be measured on a line through *A*, perpendicular to $y = 2x + 3$.

I called the point where the red line and the blue line intersect point *B*.

I had to calculate the length of *AB*. To do this, I needed the coordinates of *B*, which meant that I needed the equation of the red line.

The slope of $y = 2x + 3$ is 2.
The slope of *AB* is $-\frac{1}{2}$.

Since the red line is perpendicular to the blue line, the slope of *AB* is the negative reciprocal of 2.

Therefore, $y = -\frac{1}{2}x + b$ is an equation for the red line.

$5 = -\frac{1}{2}(6) + b$
$5 = -3 + b$
$8 = b$

I substituted the coordinates of *A* into the equation to determine *b*.

Therefore, $y = -\frac{1}{2}x + 8$ is the equation of the red line.

$$y = 2x + 3$$
$$y = -\frac{1}{2}x + 8$$

To determine the point of intersection, I had to solve a system of equations.

$$-\frac{1}{2}x + 8 = 2x + 3$$

I used substitution to replace y in the first equation with the right side of the second equation. Then I solved for x.

$$2\left(-\frac{1}{2}\right)x + 2(8) = 2(2x) + 2(3)$$
$$-x + 16 = 4x + 6$$
$$16 - 6 = 4x + x$$
$$10 = 5x$$
$$2 = x$$

$$y = 2(2) + 3$$
$$y = 7$$

I let $x = 2$ in the first equation to determine the value of y.

The coordinates of B are (2, 7).

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$
$$d_{AB} = \sqrt{(2 - 6)^2 + (7 - 5)^2}$$
$$= \sqrt{(-4)^2 + 2^2}$$
$$= \sqrt{20}$$
$$\doteq 4.5$$

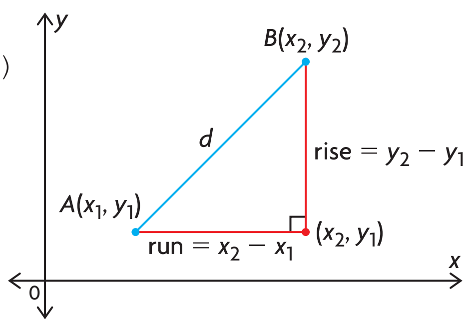
I used the distance formula to calculate the length of AB, where $A(x_1, y_1) = A(6, 5)$ and $B(x_2, y_2) = B(2, 7)$.

The point (6, 5) is about 4.5 units away from the line $y = 2x + 3$.

In Summary

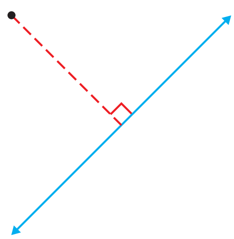
Key Idea

- The distance, d, between the endpoints of a line segment, $A(x_1, y_1)$ and $B(x_2, y_2)$, can be calculated using the distance formula:
  $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$



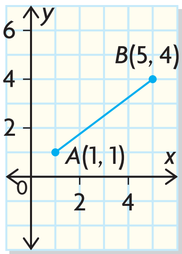
Need to Know

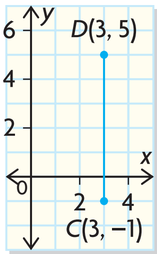
- The Pythagorean theorem is used to develop the distance formula, by calculating the straight-line distance between two points.
- The distance between a point and a line is the shortest distance between them. It is measured on a perpendicular line from the point to the line.

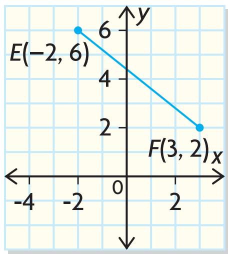


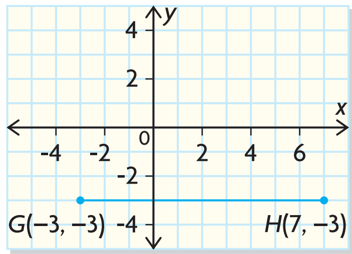
CHECK Your Understanding

1. Determine the length of each line segment.

a) 

b) 

c) 

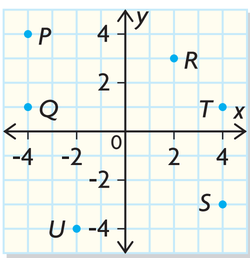
d) 

2. For each pair of points:
   i) Draw the line segment joining the points.
   ii) Determine the length of the line segment.

   a) $P(-4, 4)$ and $Q(3, 1)$
   b) $R(2, -1)$ and $S(10, 2)$
   c) $T(3.5, -3)$ and $U(3.5, 11)$
   d) $X(-1, 6)$ and $Y(5, 6)$

3. A helicopter travelled from Kapuskasing to North Bay. On a map of Ontario, with the origin at Windsor and the coordinates in kilometres, Kapuskasing is at $K(-70, 770)$ and North Bay is at $N(220, 490)$.
   a) Approximately how far did the helicopter travel?
   b) What assumption did you make about the route of the helicopter?

PRACTISING



4. Calculate the distance between each pair of points in the diagram at the left.
   a) *P* and *Q*
   b) *Q* and *R*
   c) *U* and *S*
   d) *P* and *R*
   e) *P* and *U*
   f) *Q* and *T*

5. For each pair of points below:
   i) Draw the line segment joining the points.
   ii) Calculate the length of the line segment.

   a) $A(2, 6)$ and $B(5, 2)$
   b) $C(-3, 4)$ and $D(3, 2)$
   c) $E(-6, 8)$ and $F(-6, -9)$
   d) $G(0, -7)$ and $H(1, 3)$
   e) $I(-3, -3)$ and $J(5, -4)$
   f) $K(-10, -2)$ and $L(6, -2)$

6. a) Which line segment(s) for question 5 are vertical? Which are horizontal? Explain how you know.
   b) How can you calculate the length of a vertical or horizontal line segment without using the distance formula?

7. A coordinate system is superimposed on a billiard table. Gord has a yellow ball at $A(2, 3)$. He is going to "bank" it off the side rail at $B(6, 5)$, into the pocket at $C(2, 7)$. How far will the yellow ball travel?

8. **K** Which of these points is closest to point $A(-3.2, 5.6)$: $B(1.8, -4.3)$, $C(0.7, 8.9)$, or $D(-7.6, 3.9)$? Justify your decision.

9. A forest fire is threatening two small towns, Mordon and Bently. On a map, the fire is located at (10, −11), the fire hall in Mordon is located at (26, 77), and the fire hall in Bently is located at (12, −88). Which fire hall is closer to the fire?

10. In a video game, three animated characters are programmed to run out of a building at $F(1, -1)$ and head in three different directions. After 2 s, Animal is at $A(22, 18)$, Beast is at $B(-3, 35)$, and Creature is at $C(7, -29)$. Which character ran farthest?

11. **C** How are the formulas for calculating the length of a line segment and the midpoint of a line segment, using the coordinates of the endpoints, the same? How are they different?

12. Calculate the distance between each line and the point. Round your answer to one decimal place.

a) $y = 4x - 2$, $(-3, 3)$

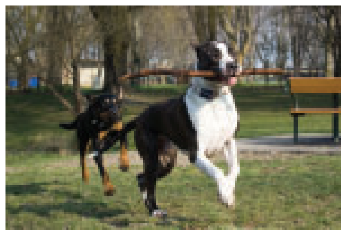
b) $y = -x + 5$, $(-1, -2)$

c) $2x + 3y = 6$, $(7, 6)$

d) $5x - 2y = 10$, $(2, 4.5)$

13. **T** A new amusement park is going to be built near two major highways. On a coordinate grid of the area, with the scale 1 unit represents 1 km, the park is located at $P(3, 4)$. Highway 2 is represented by the equation $y = 2x + 5$, and Highway 10 is represented by the equation $y = -0.5x + 2$. Determine the coordinates of the exits that must be built on each highway to result in the shortest road to the park.

14. A coordinate grid is superimposed on the plan of a new housing development. A fibre-optic cable is being laid to link points $A(-18, 12)$, $B(-8, 1)$, $C(3, 4)$, and $D(15, 7)$ in a run beginning at A and ending at D. If one unit on the grid represents 2.5 m, how much cable is required?



15. **A** A leash-free area for dogs is going to be created in a field behind a recreation centre. The area will be in the shape of an irregular pentagon, with vertices at (2, 0), (1, 6), (8, 9), (10, 7), and (6, 0). If one unit on the plan represents 10 m, what length of fencing will be required?

16. Suppose that you know the coordinates of three points. Explain how you would determine which of the first two points is closer to the third point. Describe the procedures, facts, and formulas you would use, and give an example.

Extending

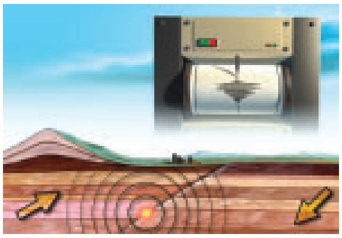
17. $\triangle ABC$ has vertices at $A(1, 2)$, $B(4, 8)$, and $C(8, 4)$.

a) $\triangle ABC$ is translated so that vertex A' is on the x-axis and vertex B' is on the y-axis. Determine the coordinates of the translated triangle, $\triangle A'B'C'$.

b) $\triangle DEF$ has vertices at $D(-1, 1)$, $E(-2, 6)$, and $F(-8, 3)$. Is $\triangle DEF$ congruent to $\triangle ABC$? Justify your answer.

2.3 Equation of a Circle

YOU WILL NEED
- graphing calculator
- grid paper, ruler, and compass, or dynamic geometry software



Career Connection

A geologist studies the physical structure and processes of Earth. Professional geologists work for a wide range of government agencies, private firms, non-profit organizations, and academic instutions.

Tech Support

For help constructing a circle and plotting points using dynamic geometry software, see Appendix B-34 and B-18.

GOAL

Develop and use an equation for a circle.

INVESTIGATE the Math

When an earthquake occurs, seismographs can be used to record the shock waves. The shock waves are then used to locate the epicentre of the earthquake—the point on Earth located directly above the rock movement. The time lag between the shock waves is used to calculate the distance between the epicentre and each recording station, which avoids considering direction.

A seismograph in Collingwood, Ontario, recorded vibrations indicating that the epicentre of an earthquake was 30 km away.

? What equation describes the possible locations of the epicentre of the earthquake, if (0, 0) represents the location of the seismograph?

A. Tell why the equation of a circle that has its centre at the origin and a radius of 30 describes all the possible locations of the epicentre.

B. Sketch this circle on a grid. Then identify the coordinates of all its intercepts.

C. Show that (24, 18), (−24, 18), (24, −18), and (−24, −18) are possible locations of the epicentre, using
i) your graph **ii)** the distance formula

D. Let point $A(x, y)$ be any possible location of the epicentre. What is the length of OA? Explain.

E. Use the distance formula to write an expression for the length of OA.

F. Use your results for parts D and E to write an equation for the circle that is centred at the origin. Write your equation in a form that does not contain a square root. Explain why your equation describes all the possible locations of the epicentre.

Reflecting

G. If (x, y) is on the circle that is centred at the origin, so are $(x, -y)$, $(-x, -y)$, and $(-x, y)$. How does your equation show this?

H. How is the equation of a circle different from the equation of a linear relationship?

I. What is the equation of any circle that has its centre at the origin and a radius of r units?

APPLY the Math

EXAMPLE 1 **Selecting a strategy** to determine the equation of a circle

A stone is dropped into a pond, creating a circular ripple. The radius of the ripple increases by 4 cm/s. Determine an equation that models the circular ripple, 10 s after the stone is dropped.

Aurora's Solution

$$x^2 + y^2 = r^2$$
$$r = (4\text{ cm/s})(10\text{ s})$$
$$r = 40\text{ cm}$$

I named the point where the stone entered the water (0, 0). I knew that the equation of a circle with centre (0, 0) and radius r is $x^2 + y^2 = r^2$.

I wanted to determine the radius of the circle at 10 s, so I multiplied the rate at which the radius increases by 10.

$$x^2 + y^2 = 40^2$$
$$x^2 + y^2 = 1600$$

I substituted the value of the radius for r into the equation.

The equation of the circular ripple is $x^2 + y^2 = 1600$.

EXAMPLE 2 **Selecting a strategy** to graph a circle, given the equation of the circle

A circle is defined by the equation $x^2 + y^2 = 25$. Sketch a graph of this circle.

Francesco's Solution

$$x^2 + y^2 = 25$$

To determine the x-intercepts, let $y = 0$.

I decided to determine some points on the circle. I knew that I could determine the intercepts by setting each variable equal to 0 and solving for the other variable.

$$x^2 + 0^2 = 25$$
$$x^2 = 25$$
$$\sqrt{x^2} = \pm\sqrt{25}$$
$$x = \pm 5$$

I remembered that there are two possible square roots: one positive and one negative.

The x-intercepts are located at (5, 0) and (−5, 0).

To determine the y-intercepts, let $x = 0$.

$$0^2 + y^2 = 25$$
$$y^2 = 25$$
$$\sqrt{y^2} = \pm\sqrt{25}$$
$$y = \pm 5$$

I solved for y.

The y-intercepts are located at (0, 5) and (0, −5).

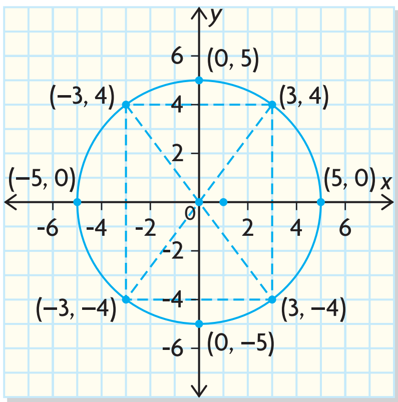
$$M = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$
$$M_{AB} = \left(\frac{5 + (-5)}{2}, \frac{0 + 0}{2}\right)$$
$$= (0, 0)$$

The centre is at (0, 0).

Since a circle has symmetry, I reasoned that the x-intercepts are endpoints of a diameter. Since all the points on a circle are the same distance from the centre, the midpoint of this diameter must be the centre.

The radius equals 5 units.

The line segments that join (0, 0) to each intercept are radii. Since these are horizontal and vertical lines whose lengths are 5 units, this circle has a radius of 5 units.



I plotted the intercepts and then joined them with a smooth circle. I noticed that (3, 4) is on the circle, and that points with similar coordinates are too. This makes sense because a circle with centre (0, 0) is symmetrical about any line that passes through the origin.

EXAMPLE 3 Reasoning to determine the equation of a circle

A circle has its centre at (0, 0) and passes through point (8, −6).

a) Determine the equation of the circle.

b) Determine the other endpoint of the diameter through (8, −6).

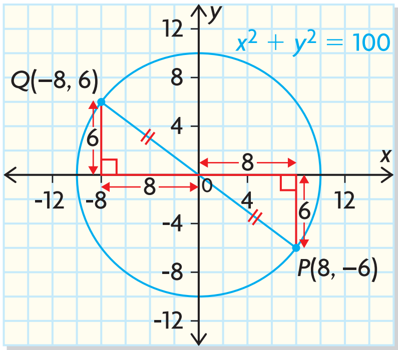
Trevor's Solution

a)
$$x^2 + y^2 = r^2$$
$$8^2 + (-6)^2 = r^2$$
$$100 = r^2$$

The equation of the circle is $x^2 + y^2 = 100$.

I started with the equation $x^2 + y^2 = r^2$, since the circle is centred at the origin. I knew that point (8, −6) is on the circle, so I substituted 8 for x and −6 for y into the equation.

b) $r = \sqrt{100}$
$r = 10$



I calculated the radius of the circle. Then I drew a sketch, making sure that the circle passed through 10 and −10 on both the *x*- and *y*-axes since 10 is the radius.

I drew the diameter that starts at *P*(8, −6), passes through (0, 0), and ends at point *Q*.

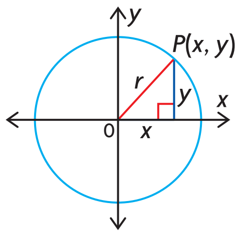
Because a circle is symmetrical and *PQ* is a diameter, I reasoned that *Q* has coordinates (−8, 6).

The other endpoint of the diameter that passes through point (8, −6) has coordinates (−8, 6).

In Summary

Key Idea

- Using the distance formula, you can show that the equation of a circle with centre (0, 0) and radius *r* is $x^2 + y^2 = r^2$.



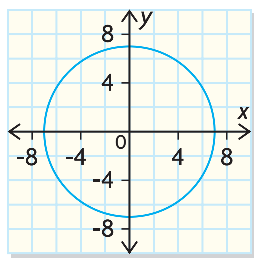
Need to Know

- Every point on the circumference of a circle is the same distance from the centre of the circle.
- Once you know one point on a circle with centre (0, 0), you can determine other points on the circle using symmetry. If (x, y) is on a circle with centre (0, 0), then so are $(-x, y)$, $(-x, -y)$, and $(x, -y)$.

CHECK Your Understanding

1. The graph at the right shows a circle with its centre at (0, 0).

a) State the *x*-intercepts of the circle.
b) State the *y*-intercepts.
c) State the radius.
d) Write the equation of the circle.



2. Write the equation of a circle with centre (0, 0) and radius *r*.

a) $r = 3$ **b)** $r = 50$ **c)** $r = 2\frac{1}{3}$ **d)** $r = 400$ **e)** $r = 0.25$

3. For each equation of a circle:
   i) Determine the radius.
   ii) State the x- and y-intercepts.
   iii) Sketch its graph.

   a) $x^2 + y^2 = 36$
   b) $x^2 + y^2 = 49$
   c) $x^2 + y^2 = 0.04$
   d) $x^2 + y^2 = 169$

PRACTISING

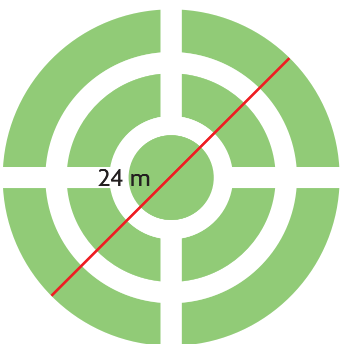
4. Use the given information to write an equation for a circle with centre (0, 0).
   a) radius of 11 units
   b) x-intercepts $(-9, 0)$ and $(9, 0)$
   c) y-intercepts $(0, -4)$ and $(0, 4)$
   d) diameter of 12 units

5. A circle has its centre at (0, 0) and passes through the point (8, 15).
   a) Calculate the radius of the circle.
   b) Write an equation for the circle.
   c) Sketch the graph.

6. Which of the following points are on the circle with equation $x^2 + y^2 = 65$? Explain.
   a) $(-4, 7)$ b) $(5, -6)$ c) $(8, -1)$ d) $(-3, -6)$

7. **K** a) Determine the radius of a circle that is centred at (0, 0) and passes through
   i) $(-3, 4)$ ii) $(5, 0)$ iii) $(0, -3)$ iv) $(8, -15)$
   b) Write the equation of each circle in part a).
   c) State the coordinates of two other points on each circle.
   d) Sketch the graph of each circle.

8. Write an equation for a circle that models each situation. Assume that (0, 0) is the centre of the circle in each situation.
   a) the possible locations of the epicentre of an earthquake, which is recorded to be a distance of 144 km from a seismograph station in Toronto
   b) the path of a satellite in a circular orbit at a distance of 19 000 km from the centre of Earth
   c) the rim of a bicycle wheel with a diameter of 69 cm
   d) the cross-section of a storm-water tunnel that has a diameter of 2.4 m



9. Don has designed a circular vegetable garden with a diameter of 24.0 m, as shown in the sketch at the left. He has included a circular flowerbed, 6.0 m in diameter, at the centre of the garden, as well as paths that are 1.5 m wide. Don needs to make a plan on grid paper for the landscape gardeners, who will create the garden. Determine the equations of all the circles he needs to draw. Assume that the centre of the garden is (0, 0) and all noncircular gardens are of equal width.

10. **A** Two satellites are orbiting Earth. The path of one satellite has the equation $x^2 + y^2 = 56\ 250\ 000$. The orbit of the other satellite is 200 km farther from the centre of Earth. In one orbit, how much farther does the second satellite travel than the first satellite?

Career Connection

Engineers design satellites for communication including television, the Internet, and phone systems. Other uses of satellites include observation in the area of espionage, geology, and navigation such as GPS systems.

11. A circle has its centre at (0, 0) and passes through point $P(5, -12)$.
- **a)** Determine the equation of the circle.
- **b)** Determine the coordinates of the other endpoint of the diameter that passes through point P.

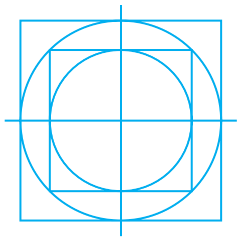
12. Determine the equation of a circle that has a diameter with endpoints $(-8, 15)$ and $(8, -15)$.

13. A rock is dropped into a pond, creating a circular ripple. The radius of the ripple increases steadily at 6 cm/s. A toy boat is floating on the pond, 2.00 m east and 1.00 m north of the spot where the rock is dropped. How long does the ripple take to reach the boat?

14. Points $(a, 5)$ and $(9, b)$ are on the circle $x^2 + y^2 = 125$. Determine the possible values of a and b. Round to one decimal place, if necessary.

15. **C** A satellite orbits Earth on a path with $x^2 + y^2 = 45\ 000\ 000$. Another satellite, in the same plane, is currently located at (12 504, 16 050). Explain how you would determine whether the second satellite is inside or outside the orbit of the first satellite.

16. **T** Chanelle is creating a design for vinyl flooring. She uses circles and squares to create the design, as shown. If the equation of the small circle is $x^2 + y^2 = 16$, what are the dimensions of the large square?



17. What reasons would you use to convince someone that it makes sense for the equation of a circle with centre (0, 0) and radius r to be $x^2 + y^2 = r^2$? Use as many reasons as you can.

Extending

18. Describe the circle with each equation.
- **a)** $9x^2 + 9y^2 = 16$
- **b)** $(x - 2)^2 + (y + 4)^2 = 9$

19. A truck with a wide load, proceeding slowly along a secondary road, is approaching a tunnel that is shaped like a semicircle. The maximum height of the tunnel is 5.25 m. If the load is 8 m wide and 3.5 m high, will it fit through the tunnel? Show your calculations, and explain your reasoning.

2 Mid-Chapter Review

FREQUENTLY ASKED Questions

Study *Aid*

- See Lesson 2.1, Example 1.
- Try Mid-Chapter Review Questions 1 and 2.

Q: How do you determine the coordinates of the midpoint of a line segment if you know the coordinates of the endpoints?

A: You can use the midpoint formula $M = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$. This formula shows that the coordinates of the midpoint are the means of the coordinates of the endpoints.

Study *Aid*

- See Lesson 2.2, Examples 1 to 3.
- Try Mid-Chapter Review Questions 6 to 9.

Q: How do you determine the length of a line segment if you know the coordinates of the endpoints?

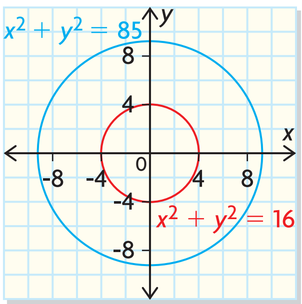
A: If the endpoints have the same x-coordinate, then the line segment is vertical. The length of the line segment is the difference in the y-coordinates of the endpoints. Similarly, if the endpoints have the same y-coordinate, then the line segment is horizontal. The length of the line segment is the difference in the x-coordinates of the endpoints.

For all types of line segments, including those which are neither vertical nor horizontal, you can use the distance formula to calculate its length.

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Study *Aid*

- See Lesson 2.3, Examples 1 and 3.
- Try Mid-Chapter Review Question 11.



Q: How do you determine the equation of a circle that has its centre at the origin?

A1: The equation of a circle with centre (0, 0) is $x^2 + y^2 = r^2$, where r is the radius. For example, the equation of a circle with centre (0, 0) and a radius of 4 units is $x^2 + y^2 = 4^2$, or $x^2 + y^2 = 16$.

A2: If you only know the coordinates of a point on the circle, you can substitute these values for x and y and then solve for r. For example, suppose that you want to determine the equation of a circle that has its centre at the origin and passes through point (2, −9). You substitute 2 for x and −9 for y.

$$2^2 + (-9)^2 = r^2$$
$$4 + 81 = r^2$$
$$85 = r^2$$

The circle has equation $x^2 + y^2 = 85$.

PRACTICE Questions

Lesson 2.1

1. Determine the coordinates of the midpoint of the line segment with each pair of endpoints.
a) $(-1, -2)$ and $(-7, 10)$
b) $(5, -1)$ and $(-2, 9)$
c) $(0, -4)$ and $(0, 12)$
d) $(6, 4)$ and $(0, 0)$

2. A diameter of a circle has endpoints $A(9, -4)$ and $B(3, -2)$. Determine the centre of the circle.

3. Describe all the points that are the same distance from points $A(-3, -1)$ and $B(5, 3)$.

4. A hockey arena is going to be built to serve two rural towns. On a plan of the area, the towns are located at $(1, 7)$ and $(8, 5)$. If the arena needs to be the same distance from both towns, determine an equation to describe the possible locations for the arena.

5. $\triangle PQR$ has vertices at $P(12, 4)$, $Q(-6, 2)$, and $R(-4, -2)$.
a) Determine the coordinates of the midpoints of its sides.
b) Determine the equation of the median from vertex Q.
c) What is the equation of the perpendicular bisector of side PQ?

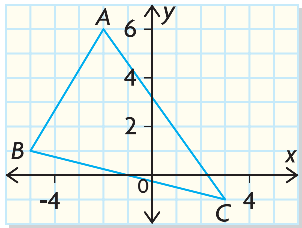
Lesson 2.2

6. Calculate the distance between each pair of points.
a) $(2, 2)$ and $(7, 4)$
b) $(-3, 0)$ and $(8, -5)$
c) $(2, 9)$ and $(-5, 9)$
d) $(9, -3)$ and $(12, -4)$

7. A power line is going to be laid from $A(-22, 15)$ to $B(7, 33)$ to $C(10, 18)$ to $D(-1, 4)$. If the units are metres, what length will the power line be?

8. Determine the distance between point $(-4, 4)$ and the line $y = 3x - 4$.

9. Show that $\triangle ABC$ has three unequal sides.



Lesson 2.3

10. **i)** State the coordinates of the centre of the circle described by each equation below.
ii) State the radius and the x- and y-intercepts of the circle.
iii) Sketch a graph of the circle.
a) $x^2 + y^2 = 169$
b) $x^2 + y^2 = 2.89$
c) $x^2 + y^2 = 98$

11. Determine the equation of a circle that has its centre at $(0, 0)$ and passes through each point.
a) $(-5, 0)$
b) $(0, 7)$
c) $(-3, -8)$
d) $(4, -9)$

12. A raindrop falls into a puddle, creating a circular ripple. The radius of the ripple grows at a steady rate of 5 cm/s. If the origin is used as the location where the raindrop hits the puddle, determine the equation that models the ripple exactly 6 s after the raindrop hits the puddle.

13. Determine whether each point is on, inside, or outside the circle $x^2 + y^2 = 45$. Explain your reasoning.
a) $(6, -3)$
b) $(-1, 7)$
c) $(-3, 5)$
d) $(-7, -2)$

14. A line segment has endpoints $A(6, -7)$ and $B(2, 9)$.
a) Verify that the endpoints of AB are on the circle with equation $x^2 + y^2 = 85$.
b) Determine the equation of the perpendicular bisector of AB.
c) Explain how you can tell, from its equation, that the perpendicular bisector goes through the centre of the circle.

2.4 Classifying Figures on a Coordinate Grid

YOU WILL NEED

- grid paper and ruler, or dynamic geometry software

GOAL

Use properties of line segments to classify two-dimensional figures.

LEARN ABOUT the Math

A surveyor has marked the corners of a lot where a building is going to be constructed. The corners have coordinates $P(-5, -5)$, $Q(-30, 10)$, $R(-5, 25)$, and $S(20, 10)$. Each unit represents 1 m. The builder wants to know the perimeter and shape of this building lot.

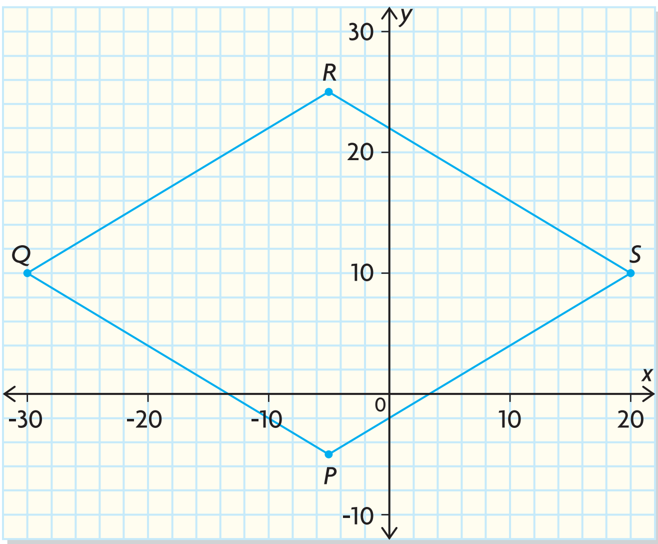
? How can the builder use the coordinates of the corners to determine the shape and perimeter of the lot?

EXAMPLE 1 Connecting slopes and lengths of line segments to classifying a figure

Use **analytic geometry** to identify the shape of quadrilateral *PQRS* and its perimeter.

analytic geometry
geometry that uses the *xy*-axes, algebra, and equations to describe relations and positions of geometric figures

Anita's Solution



I plotted the points on grid paper and then joined them to draw the figure.

I saw that the shape of the building lot looked like a parallelogram or a rhombus, but I couldn't be sure.

I knew that if *PQRS* was either of these figures, the opposite sides would be parallel. I also knew that if *PQRS* was a rhombus, all the sides would be the same length.

$$\text{Length} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

$$PQ = \sqrt{[-30 - (-5)]^2 + [10 - (-5)]^2}$$
$$= \sqrt{625 + 225}$$
$$= \sqrt{850}$$
$$\doteq 29.15$$

The length of PQ is about 29.15 units.

$$QR = \sqrt{[-5 - (-30)]^2 + (25 - 10)^2}$$
$$= \sqrt{625 + 225}$$
$$= \sqrt{850}$$
$$\doteq 29.15$$

The length of QR is about 29.15 units.

$$RS = \sqrt{[20 - (-5)]^2 + (10 - 25)^2}$$
$$= \sqrt{625 + 225}$$
$$= \sqrt{850}$$
$$\doteq 29.15$$

The length of RS is about 29.15 units.

$$SP = \sqrt{(-5 - 20)^2 + (-5 - 10)^2}$$
$$= \sqrt{625 + 225}$$
$$= \sqrt{850}$$
$$\doteq 29.15$$

The length of SP is about 29.15 units.

$$\text{Slope} = \frac{y_2 - y_1}{x_2 - x_1}$$

$$m_{PQ} = \frac{10 - (-5)}{-30 - (-5)} = \frac{15}{-25} = -\frac{3}{5}$$

$$m_{QR} = \frac{25 - 10}{-5 - (-30)} = \frac{15}{25} = \frac{3}{5}$$

$$m_{RS} = \frac{10 - 25}{20 - (-5)} = \frac{-15}{25} = -\frac{3}{5}$$

$$m_{SP} = \frac{-5 - 10}{-5 - 20} = \frac{-15}{-25} = \frac{3}{5}$$

I decided to calculate the slope and the length of each side of *PQRS*.

The slopes of *PQ* and *RS* are the same, so they are parallel. The slopes of *QR* and *SP* are also the same, so they are parallel too.

My length calculations showed that all four sides are equal.

$PQ \parallel RS$ and $QR \parallel SP$

$PQ = QR = RS = SP$

Since the opposite sides are parallel and all the side lengths are equal, *PQRS* is a rhombus.

$$\text{Perimeter} \doteq 4(29.15)$$
$$= 116.6$$

I multiplied the side length by 4 to calculate the perimeter.

The building lot is a rhombus. Its perimeter measures about 116.6 m.

Communication *Tip*

The symbol || is used to replace the phrase "is parallel to."

Reflecting

A. Why could Anita not rely completely on her diagram to determine the shape of the quadrilateral?

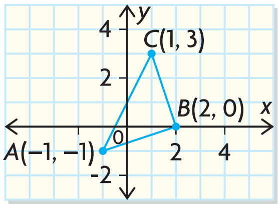
B. Why did Anita need to calculate the slopes and lengths of all the sides to determine the shape of the building lot? Explain.

APPLY the Math

EXAMPLE 2 | **Reasoning** about lengths and slopes to classify a triangle

A triangle has vertices at $A(-1, -1)$, $B(2, 0)$, and $C(1, 3)$. What type of triangle is it? Justify your decision.

Angelica's Solution



I drew a diagram of the triangle on grid paper. I thought that the triangle might be isosceles since *AB* and *BC* look like they are the same length. The triangle might also be a right triangle since $\angle ABC$ looks like it might be 90°.

$$\begin{aligned} AB &= \sqrt{[2-(-1)]^2 + [0-(-1)]^2} \\ &= \sqrt{3^2 + 1^2} \\ &= \sqrt{10} \\ &\doteq 3.2 \\ BC &= \sqrt{(1-2)^2 + (3-0)^2} \\ &= \sqrt{(-1)^2 + 3^2} \\ &= \sqrt{10} \\ &\doteq 3.2 \end{aligned}$$

To determine the type of triangle, I had to know the lengths of the sides. To check my isosceles prediction, I used the distance formula to determine the lengths of *AB* and *BC*. I rounded each answer to one decimal place.

AB and *BC* are the same length, so $\triangle ABC$ is isosceles.

$$\text{Slope} = \frac{y_2 - y_1}{x_2 - x_1}$$

$$\begin{aligned} m_{AB} &= \frac{0-(-1)}{2-(-1)} \\ &= \frac{1}{3} \\ m_{BC} &= \frac{3-0}{1-2} \\ &= \frac{3}{-1} \\ &= -3 \end{aligned}$$

To determine if the triangle has a right angle, I had to determine if a pair of sides are perpendicular. I did this by comparing their slopes. I calculated the slopes of the sides that looked perpendicular, *AB* and *BC*.

The slopes of *AB* and *BC* are negative reciprocals, so $AB \perp BC$. This means that $\triangle ABC$ is a right triangle.

$\triangle ABC$ is an isosceles right triangle, with $AB = BC$ and $\angle ABC = 90°$.

Communication *Tip*

The symbol $\perp$ is used to replace the phrase "is perpendicular to."

EXAMPLE 3 Solving a problem using properties of line segments

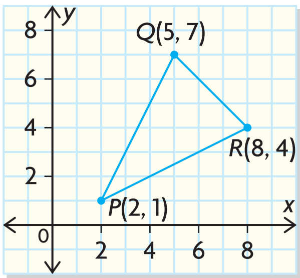
Tony is constructing a patterned concrete patio that is in the shape of an isosceles triangle, as requested by his client. On his plan, the vertices of the triangle are at $P(2, 1)$, $Q(5, 7)$, and $R(8, 4)$. Each unit represents 1 m.

a) Confirm that the plan shows an isosceles triangle.

b) Calculate the area of the patio.

Tony's Solution

a)



I made a sketch of the triangle. It looks isosceles since *PQ* and *RP* appear to be the same length.

$$\text{Length} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

$$PQ = \sqrt{(5 - 2)^2 + (7 - 1)^2}$$
$$= \sqrt{45}$$
$$\doteq 6.7$$

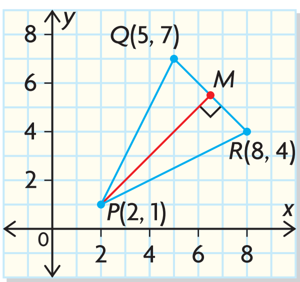
I used the distance formula to calculate the lengths of the sides of the triangle. I saw that *PQ* is the same length as *RP*, so the triangle is isosceles.

$$RP = \sqrt{(2 - 8)^2 + (1 - 4)^2}$$
$$= \sqrt{45}$$
$$\doteq 6.7$$

$$QR = \sqrt{(8 - 5)^2 + (4 - 7)^2}$$
$$= \sqrt{18}$$
$$\doteq 4.2$$

Since $PQ = RP$, the triangle is isosceles.

b)



I knew that I needed the lengths of the base and height to calculate the area of the triangle, since $\text{area} = \dfrac{\text{base} \times \text{height}}{2}$.

I remembered that, in an isosceles triangle, the median from the vertex where the two equal sides meet is perpendicular to the side that is opposite this vertex. *PM* is the height of the triangle and *QR* is its base.

Midpoint of QR is

$$M = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$
$$= \left(\frac{5 + 8}{2}, \frac{7 + 4}{2}\right)$$
$$= (6.5, 5.5)$$

I calculated M, the midpoint of QR, so I could use it to determine the length of PM.

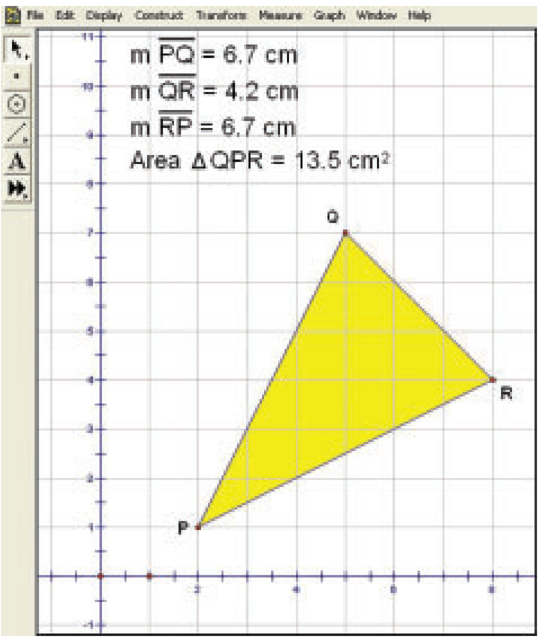
$$PM = \sqrt{(6.5 - 2)^2 + (5.5 - 1)^2}$$
$$= \sqrt{20.25 + 20.25}$$
$$= \sqrt{40.5}$$
$$\doteq 6.4$$

I used the distance formula to calculate the length of PM. I already knew that the length of QR is $\sqrt{18}$. PM is the height of the triangle, and QR is the base.

$$\text{Area of } \triangle PQR = \frac{QR \times PM}{2}$$
$$= \frac{\sqrt{18} \times \sqrt{40.5}}{2}$$
$$= 13.5$$

I calculated the area of the triangle using the exact values to minimize the rounding error.

The triangular patio has an area of 13.5 m^2.



I checked my calculations by plotting the vertices of the triangle using dynamic geometry software. Then I constructed the triangle and its interior.

I measured the lengths of the sides and determined the area. The scale in this sketch is 1 unit = 1 cm instead of 1 unit = 1 m. Since the numbers were the same, however, I knew that my calculations were correct.

Tech Support

Do not change the scale of the dynamic geometry software grid, because this will make the unit length different from 1 cm.

In Summary

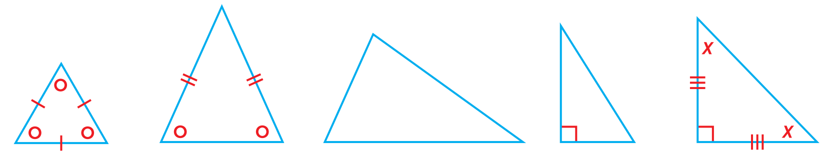
Key Idea

- When a geometric figure is drawn on a coordinate grid, the coordinates of its vertices can be used to calculate the slopes and lengths of the line segments, as well as the coordinates of the midpoints.

Need to Know

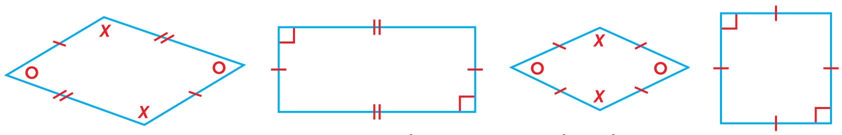
- Triangles and quadrilaterals can be classified by the relationships between their sides and their interior angles.

Triangles

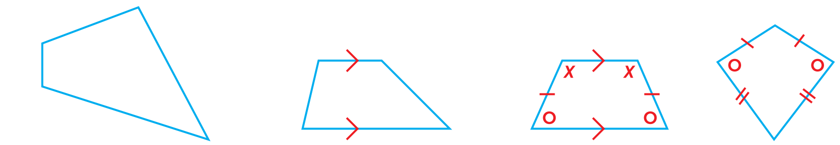


equilateral triangle | isosceles triangle | scalene triangle | right triangle | isosceles right triangle

Quadrilaterals



parallelogram | rectangle | rhombus | square



irregular quadrilateral | trapezoid | isosceles trapezoid | kite

- To solve a problem that involves a geometric figure, it is a good idea to start by drawing a diagram of the situation on a coordinate grid.
- Parallel lines have the same slope.
- Perpendicular lines have slopes that are negative reciprocals.

CHECK Your Understanding

Round all answers to two decimal places, where necessary.

1. Show that the line segment joining points $P(1, 4)$ and $Q(5, 5)$ is parallel to the line segment joining points $R(3, -4)$ and $S(7, -3)$.

2. Show that TU, $T(-1, 7)$ and $U(3, 5)$, is perpendicular to VW, $V(-4, 1)$ and $W(-1, 7)$.

3. The sides of quadrilateral *ABCD* have the following slopes.

Side	*AB*	*BC*	*CD*	*AD*
Slope	-5	$-\frac{1}{7}$	-5	$-\frac{1}{7}$

What types of quadrilateral could *ABCD* be? What other information is needed to determine the exact type of quadrilateral?

4. $\triangle DEF$ has vertices at $D(-3, -4)$, $E(-2, 4)$, and $F(5, -5)$.
   a) Show that $\triangle DEF$ is isosceles.
   b) Determine the length of the median from vertex *D*.
   c) Show that this median is perpendicular to *EF*.

PRACTISING

5. The lengths of the sides in a quadrilateral are $PQ = 4.5$ units, $QR = 4.5$ units, $RS = 4.5$ units, and $SP = 4.5$ units. What types of quadrilateral could *PQRS* be? What other information is needed to determine the exact type of quadrilateral?

6. The following points are the vertices of triangles. Predict whether each triangle is scalene, isosceles, or equilateral. Then draw the triangle on a coordinate grid and calculate each side length to check your prediction.
   a) $A(3, 3)$, $B(-1, 2)$, $C(0, -2)$
   b) $G(-1, 3)$, $H(-2, -2)$, $I(2, 0)$
   c) $D(2, -3)$, $E(-2, -4)$, $F(6, -6)$
   d) $J(2, 5)$, $K(5, -2)$, $L(-1, -2)$

7. $P(-7, 1)$, $Q(-8, 4)$, and $R(-1, 3)$ are the vertices of a triangle. Show that $\triangle PQR$ is a right triangle.

8. A triangle has vertices at $L(-7, 0)$, $M(2, 1)$, and $N(-3, 5)$. Verify that it is a right isosceles triangle.

9. **a)** How can you use the distance formula to decide whether points $P(-2, -3)$, $Q(4, 1)$, and $R(2, 4)$ form a right triangle? Justify your answer.
   b) Without drawing any diagrams, explain which sets of points are the vertices of right triangles.
      i) $S(-2, 2)$, $T(-1, -2)$, $U(7, 0)$
      ii) $X(3, 2)$, $Y(1, -2)$, $Z(-3, 6)$
      iii) $A(5, 5)$, $B(3, 8)$, $C(8, 7)$

10. A quadrilateral has vertices at $W(-3, 2)$, $X(2, 4)$, $Y(6, -1)$, and $Z(1, -3)$.
   a) Determine the length and slope of each side of the quadrilateral.
   b) Based on your calculations for part a), what type of quadrilateral is *WXYZ*? Explain.
   c) Determine the difference in the lengths of the two diagonals of *WXYZ*.

11. A polygon is defined by points $R(-5, 1)$, $S(5, 3)$, $T(2, -1)$, and $U(-8, -3)$. Show that the polygon is a parallelogram.

12. A quadrilateral has vertices at $A(-2, 3)$, $B(-2, -2)$, $C(2, 1)$, and $D(2, 6)$. Show that the quadrilateral is a rhombus.

13. K **a)** Show that $EFGH$, with vertices at $E(-2, 3)$, $F(2, 1)$, $G(0, -3)$, and $H(-4, -1)$, is a square.
b) Show that the diagonals of $EFGH$ are perpendicular to each other.

14. The vertices of quadrilateral $PQRS$ are at $P(0, -5)$, $Q(-9, 2)$, $R(-5, 8)$, and $S(4, 2)$. Show that $PQRS$ is not a rectangle.

15. C A square is a special type of rectangle. A square is also a special type of rhombus. How would you apply these descriptions of a square when using the coordinates of the vertices of a quadrilateral to determine the type of quadrilateral? Include examples in your explanation.

16. Determine the type of quadrilateral described by each set of vertices. Give reasons for your answers.
a) $J(-5, 2)$, $K(-1, 3)$, $L(-2, -1)$, $M(-6, -2)$
b) $E(-5, -4)$, $F(-5, 1)$, $G(7, 4)$, $H(7, -1)$
c) $D(-1, 3)$, $E(6, 4)$, $F(4, -1)$, $G(-3, -2)$
d) $P(-5, 1)$, $Q(3, 3)$, $R(4, -1)$, $S(-4, -3)$

17. A A surveyor is marking the corners of a building lot. If the corners have coordinates $A(-5, 4)$, $B(4, 9)$, $C(9, 0)$, and $D(0, -5)$, what shape is the building lot? Include your calculations in your answer.

18. T Points $P(4, 12)$, $Q(9, 14)$, and $R(13, 4)$ are three vertices of a rectangle.
a) Determine the coordinates of the fourth vertex, S.
b) Briefly describe how you found the coordinates of S.
c) Predict whether the lengths of the diagonals of rectangle $PQRS$ are the same length. Check your prediction.

19. Suppose that you know the coordinates of the vertices of a quadrilateral. What calculations would help you determine if the quadrilateral is a special type, such as a parallelogram, rectangle, rhombus, or square? How would you use the coordinates of the vertices in your calculations? Organize your thoughts in a flow chart.

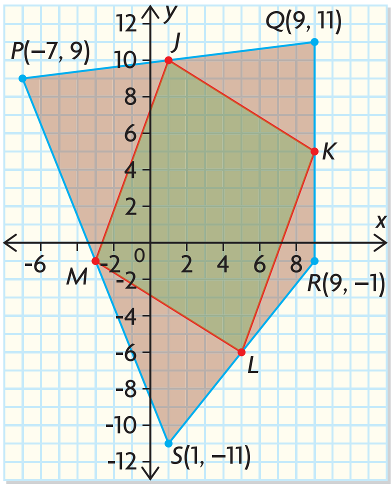
Extending

20. **a)** Show that the midpoints of any pair of sides in a triangle are two of the vertices of another triangle, which has dimensions that are exactly one-half the dimensions of the original triangle and a side that is parallel to a side in the original triangle.
b) Show that the midpoints of the sides in any quadrilateral are the vertices of a parallelogram.

2.5 Verifying Properties of Geometric Figures

YOU WILL NEED

- grid paper and ruler, or dynamic geometry software



GOAL

Use analytic geometry to verify properties of geometric figures.

LEARN ABOUT the Math

Carlos has hired a landscape designer to give him some ideas for improving his backyard, which is a quadrilateral. The designer's plan on a coordinate grid shows a lawn area that is formed by joining the midpoints of the adjacent sides in the quadrilateral. The four triangular areas will be gardens.

? How can Carlos verify that the lawn area is a parallelogram?

EXAMPLE 1 Proving a conjecture about a geometric figure

Show that the **midsegments of the quadrilateral**, with vertices at $P(-7, 9)$, $Q(9, 11)$, $R(9, -1)$, and $S(1, -11)$, form a parallelogram.

midsegment of a quadrilateral
a line segment that connects the midpoints of two adjacent sides in a quadrilateral

Ed's Solution: Using slopes

J has coordinates $\left(\frac{-7 + 9}{2}, \frac{9 + 11}{2}\right) = (1, 10)$.

K has coordinates $\left(\frac{9 + 9}{2}, \frac{11 + (-1)}{2}\right) = (9, 5)$.

L has coordinates $\left(\frac{9 + 1}{2}, \frac{-1 + (-11)}{2}\right) = (5, -6)$.

M has coordinates $\left(\frac{1 + (-7)}{2}, \frac{-11 + 9}{2}\right) = (-3, -1)$.

I used the midpoint formula to determine the coordinates of the midpoints of *PQ*, *QR*, *RS*, and *SP*, which are *J*, *K*, *L*, and *M*.

$$m_{JK} = \frac{5 - 10}{9 - 1} = -0.625 \qquad m_{LM} = \frac{-1 - (-6)}{-3 - 5} = -0.625$$

$$m_{KL} = \frac{-6 - 5}{5 - 9} = 2.75 \qquad m_{MJ} = \frac{10 - (-1)}{1 - -3} = 2.75$$

I needed to show that *JK* is parallel to *LM* and that *KL* is parallel to *MJ*.

I used the slope formula, $m = \frac{y_2 - y_1}{x_2 - x_1}$, to calculate the slopes of *JK*, *KL*, *LM*, and *MJ*.

$m_{JK} = m_{LM}$ and $m_{KL} = m_{MJ}$
$JK \parallel LM$ and $KL \parallel MJ$

Quadrilateral *JKLM* is a parallelogram.

I saw that the slopes of *JK* and *LM* are the same and the slopes of *KL* and *MJ* are the same. This means that the opposite sides in quadrilateral *JKLM* are parallel. So quadrilateral *JKLM* must be a parallelogram.

Grace's Solution: Using properties of the diagonals

J has coordinates $\left(\frac{-7 + 9}{2}, \frac{9 + 11}{2}\right) = (1, 10)$.

K has coordinates $\left(\frac{9 + 9}{2}, \frac{11 + (-1)}{2}\right) = (9, 5)$.

L has coordinates $\left(\frac{9 + 1}{2}, \frac{-1 + (-11)}{2}\right) = (5, -6)$.

M has coordinates $\left(\frac{1 + (-7)}{2}, \frac{-11 + 9}{2}\right) = (-3, -1)$.

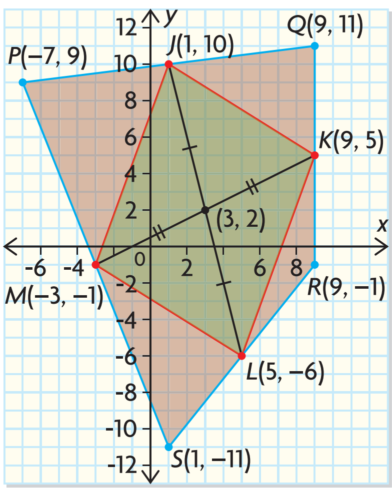
I calculated the coordinates of points *J*, *K*, *L*, and *M*, the midpoints of the sides in quadrilateral *PQRS*.

The midpoint of *JL* is $\left(\frac{1 + 5}{2}, \frac{10 + (-6)}{2}\right) = (3, 2)$.

The midpoint of *KM* is $\left(\frac{9 + (-3)}{2}, \frac{5 + (-1)}{2}\right) = (3, 2)$.

Then I calculated the midpoints of the diagonals *JL* and *KM*.

I discovered that both diagonals have the same midpoint, so they must intersect at this point.



The diagonals of the quadrilateral bisect each other since they have the same midpoint.

This means that *JKLM* must be a parallelogram.

JKLM is a parallelogram.

Reflecting

A. How is Ed's strategy different from Grace's strategy?

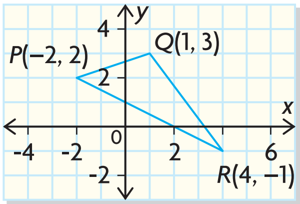
B. What is another strategy you could use to show that *JKLM* is a parallelogram?

APPLY the Math

EXAMPLE 2 Selecting a strategy to verify a property of a triangle

A triangle has vertices at $P(-2, 2)$, $Q(1, 3)$, and $R(4, -1)$. Show that the midsegment joining the midpoints of PQ and PR is parallel to QR and half its length.

Andrea's Solution: Using slopes and lengths of line segments



I drew a diagram of the triangle.

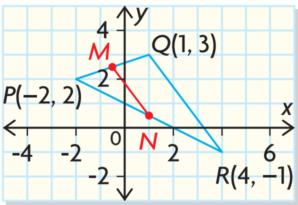
The midpoint of PQ is

$$M\left(\frac{-2 + 1}{2}, \frac{2 + 3}{2}\right) = M(-0.5, 2.5).$$

The midpoint of PR is

$$N\left(\frac{-2 + 4}{2}, \frac{2 + (-1)}{2}\right) = N(1, 0.5).$$

I determined the midpoints of PQ and PR. I used M for the midpoint of PQ and N for the midpoint of PR.



I drew the line segment that joins M to N.

I knew that the slopes of QR and MN would be the same if QR is parallel to MN.

$$m_{QR} = \frac{-1 - 3}{4 - 1}$$

$$= -\frac{4}{3}$$

$$m_{MN} = \frac{0.5 - 2.5}{1 - (-0.5)}$$

$$= \frac{-2}{1.5}$$

$$= -\frac{4}{3}$$

$MN \parallel QR$

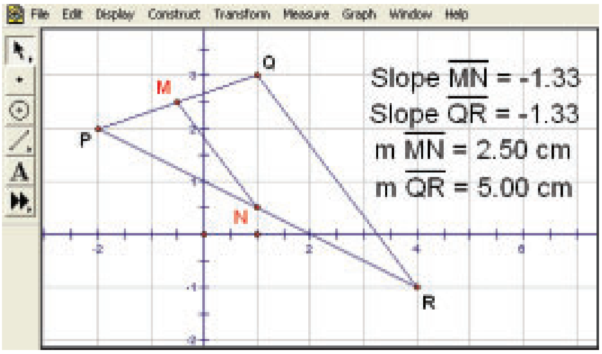
I calculated the slopes of QR and MN.

I multiplied $\frac{-2}{1.5}$ by $\frac{2}{2}$ to get $-\frac{4}{3}$. The slopes are the same, so MN is parallel to QR.

$$QR = \sqrt{(4 - 1)^2 + (-1 - 3)^2}$$
$$= \sqrt{9 + 16}$$
$$= \sqrt{25}$$
$$= 5$$
$$MN = \sqrt{[1 - (-0.5)]^2 + (0.5 - 2.5)^2}$$
$$= \sqrt{2.25 + 4}$$
$$= \sqrt{6.25}$$
$$= 2.5$$
$$MN = \frac{1}{2}QR$$

Next, I calculated the lengths of *QR* and *MN*. The length of *MN* is exactly one-half the length of *QR*.

The midsegment that joins the midpoints of *PQ* and *PR* is parallel to *QR* and one-half its length.



I verified my calculations using dynamic geometry software. I chose a scale where 1 unit = 1 cm. I constructed the triangle and the midsegment *MN*. Then I measured the lengths and slopes of *MN* and *QR*. My calculations were correct.

EXAMPLE 3 Reasoning about lines and line segments to verify a property of a circle

Show that points $A(10, 5)$ and $B(2, -11)$ lie on the circle with equation $x^2 + y^2 = 125$. Also show that the perpendicular bisector of **chord** *AB* passes through the centre of the circle.

Drew's Solution

$$r = \sqrt{125}$$
$$r \doteq 11.2$$

The intercepts are located at (0, 11.2), (0, −11.2), (11.2, 0), and (−11.2, 0).

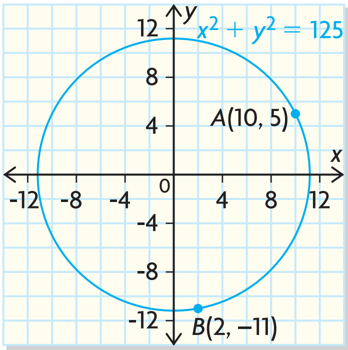
I knew that $x^2 + y^2 = 125$ is the equation of a circle with centre (0, 0) since it is in the form $x^2 + y^2 = r^2$.

I calculated the radius and used this value to determine the coordinates of the intercepts.

Left Side	Right Side	Left Side	Right Side
$x^2 + y^2$	125	$x^2 + y^2$	125
$= 10^2 + 5^2$		$= 2^2 + (-11)^2$	
$= 125$		$= 125$	

I substituted the coordinates of points *A* and *B* into the equation of the circle to show that *A* and *B* are on the circle.

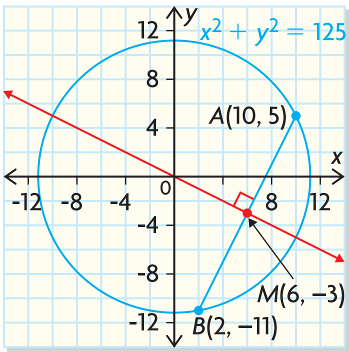
Points $A(10, 5)$ and $B(2, -11)$ lie on the circle.



I used the intercepts to sketch the circle. I marked points A and B on the circle.

The midpoint of AB is

$$M\left(\frac{10 + 2}{2}, \frac{5 + (-11)}{2}\right) = M(6, -3).$$



I determined the midpoint and marked it on my sketch. I called the midpoint M. Then I sketched the perpendicular bisector of AB.

To write an equation for the perpendicular bisector, I had to know its slope and the coordinates of a point on it. I already knew that the midpoint $M(6, -3)$ is on the perpendicular bisector.

$$m_{AB} = \frac{y_2 - y_1}{x_2 - x_1}$$
$$= \frac{-11 - 5}{2 - 10}$$
$$= \frac{-16}{-8}$$

The slope of chord AB is 2.

The slope of the perpendicular bisector is $-\frac{1}{2}$.

To determine the slope of the perpendicular bisector, I had to calculate the slope of AB.

I knew that the slope of the perpendicular bisector is the negative reciprocal of 2, which is $-\frac{1}{2}$.

An equation for the perpendicular bisector is

$$y = -\frac{1}{2}x + b.$$

Since $M(6, -3)$ lies on the perpendicular bisector,

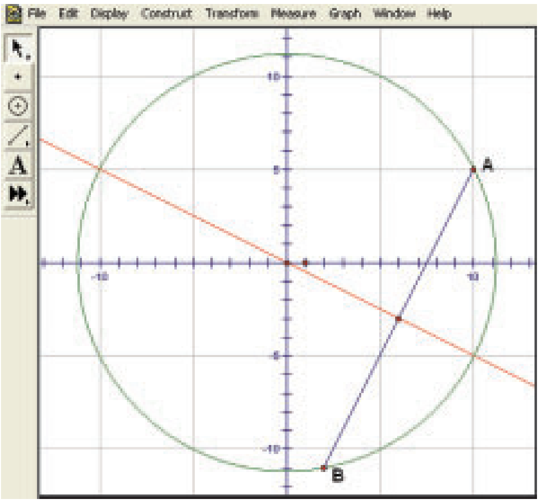
$$-3 = -\frac{1}{2}(6) + b$$
$$-3 = -3 + b$$
$$0 = b$$

I wrote the equation in the form $y = mx + b$. Then I substituted the coordinates of M into the equation to determine the value of b.

Since $b = 0$, the line goes through (0, 0), which is the centre of the circle.

The equation of the perpendicular bisector of chord AB is $y = -\frac{1}{2}x$. The y-intercept is 0.

The line passes through (0, 0), which is the centre of the circle.



I verified my calculations using dynamic geometry software. I constructed the circle, the chord, and the perpendicular bisector of the chord. The sketch confirmed that the perpendicular bisector passes through the centre of the circle.

In Summary

Key Idea

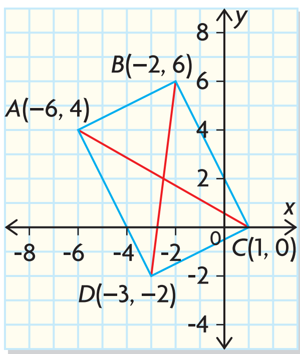
- When you draw a geometric figure on a coordinate grid, you can verify many of its properties using the properties of lines and line segments.

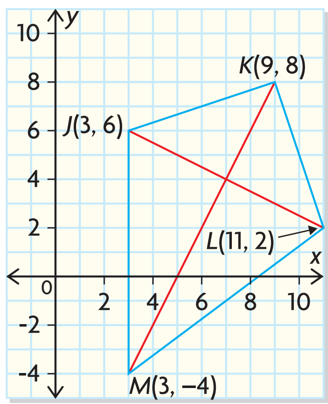
Need to Know

- You can use the midpoint formula to determine whether a point bisects a line segment.
- You can use the formula for the length of a line segment to calculate the lengths of two or more sides in a geometric figure so that you can compare them.
- You can use the slope formula to determine whether the sides in a geometric figure are parallel, perpendicular, or neither.

CHECK Your Understanding

1. Show that the diagonals of quadrilateral $ABCD$ at the right are equal in length.
2. Show that the diagonals of quadrilateral $JKLM$ at the far right are perpendicular.
3. $\triangle PQR$ has vertices at $P(-2, 1)$, $Q(1, 5)$, and $R(5, 2)$. Show that the median from vertex Q is the perpendicular bisector of PR.





PRACTISING

4. A rectangle has vertices at $J(10, 0)$, $K(-8, 6)$, $L(-12, -6)$, and $M(6, -12)$. Show that the diagonals bisect each other.

5. A rectangle has vertices at $A(-6, 5)$, $B(12, -1)$, $C(8, -13)$, and
K $D(-10, -7)$. Show that the diagonals are the same length.

6. Make a conjecture about the type of quadrilateral shown in question 1. Use
C analytic geometry to explain why your conjecture is either true or false.

7. Make a conjecture about the type of quadrilateral shown in question 2. Use analytic geometry to explain why your conjecture is either true or false.

8. A triangle has vertices at $D(-5, 4)$, $E(1, 8)$, and $F(-1, -2)$. Show that the height from D is also the median from D.

9. Show that the midsegments of a quadrilateral with vertices at $P(-2, -2)$, $Q(0, 4)$, $R(6, 3)$, and $S(8, -1)$ form a rhombus.

10. Show that the midsegments of a rhombus with vertices at $R(-5, 2)$, $S(-1, 3)$, $T(-2, -1)$, and $U(-6, -2)$ form a rectangle.

11. Show that the diagonals of the rhombus in question 10 are perpendicular and bisect each other.

12. Show that the midsegments of a square with vertices at $A(2, -12)$, $B(-10, -8)$, $C(-6, 4)$, and $D(6, 0)$ form a square.

13. **a)** Show that points $A(-4, 3)$ and $B(3, -4)$ lie on $x^2 + y^2 = 25$.
    b) Show that the perpendicular bisector of chord AB passes through the centre of the circle.

14. A trapezoid has vertices at $A(1, 2)$, $B(-2, 1)$, $C(-4, -2)$, and $D(2, 0)$.
A **a)** Show that the line segment joining the midpoints of BC and AD is parallel to both AB and DC.
    b) Show that the length of this line segment is half the sum of the lengths of the parallel sides.

15. $\triangle ABC$ has vertices at $A(3, 4)$, $B(-2, 0)$, and $C(5, 0)$. Prove that the
T area of the triangle formed by joining the midpoints of $\triangle ABC$ is one-quarter the area of $\triangle ABC$.

16. Naomi claims that the midpoint of the hypotenuse of a right triangle is the same distance from each vertex of the triangle. Create a flow chart that summarizes the steps you would take to verify this property.

Extending

17. Show that the intersection of the line segments joining the midpoints of the opposite sides of a square is the same point as the midpoints of the diagonals.

2.6 Exploring Properties of Geometric Figures

GOAL

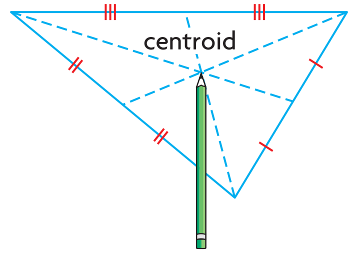
Investigate intersections of lines or line segments within triangles and circles.

YOU WILL NEED

- grid paper and ruler, or dynamic geometry software

EXPLORE the Math

When Lucy drew the medians in a triangle, she noticed that they intersected at the same point. She recalled that this is the **centroid**, or balance point, which is the centre of mass for the triangle.



? What other properties of triangles and circles are determined by the intersection of lines?

Investigating Triangles

A. Construct a triangle, and label the vertices A, B, and C. Construct the perpendicular bisector of each side of $\triangle ABC$.

B. Label the intersection of the perpendicular bisectors O. This is the **circumcentre** of $\triangle ABC$. Construct a circle with its centre at O and radius OA. What do you notice?

C. Repeat parts A and B for other triangles, including some obtuse triangles. Is the result always the same? Explain.

D. Construct a new triangle, and draw the **altitude** from each vertex. The intersection of the altitudes is the **orthocentre** of the triangle.

E. Repeat part D for other triangles. Is the intersection of the altitudes always inside the triangle? Explain.

F. Copy and complete this table to summarize what you know about intersecting lines in triangles.

Type of Triangle Centre	Type of Intersecting Lines	Special Property	Diagram
centroid			
circumcentre			
orthocentre			

circumcentre
the centre of the circle that passes through all three vertices of a triangle; the circumcentre is the same distance from all three vertices

altitude
a line segment that represents the height of a polygon, drawn from a vertex of the polygon perpendicular to the opposite side

orthocentre
the point where the three altitudes of a triangle intersect

Investigating Circles

G. Construct a circle, and draw two chords, *JK* and *LM*, that intersect inside the circle. Label the intersection point *O*. Measure the lengths of *JO*, *OK*, *LO*, and *OM*. Calculate the products $JO \times OK$ and $LO \times OM$. Comment on your results.

H. Repeat part G for other pairs of chords and other circles. Include some examples with chords that intersect outside the circle.

I. Copy and complete this table to summarize what you know about intersecting lines in circles.

Location of Intersection	Property	Diagram
inside the circle		
outside the circle		

Reflecting

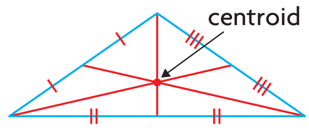
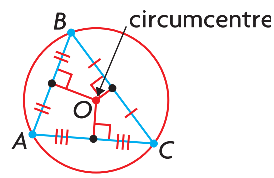
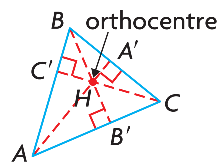
J. In what type of triangle do the medians, perpendicular bisectors, and altitudes coincide?

K. Does the size of a circle or the location of the intersecting chords affect the property you observed? Explain.

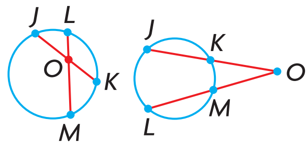
In Summary

Key Idea

- Some properties of two-dimensional figures are determined by the intersection of lines.

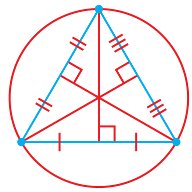
Figure	Properties Determined by the Intersection of Lines or Line Segments	Diagram
triangle	All the medians intersect at the same point, called the centroid.	
	All the perpendicular bisectors intersect at the same point, called the circumcentre. The three vertices of the triangle are the same distance from this point.	
	All the altitudes intersect at the same point, called the orthocentre.	

(continued)

circle	When two chords intersect, the products of their segments are equal.	 $JO \times OK = LO \times OM$

Need to Know

- In an equilateral triangle, the medians, perpendicular bisectors, and altitudes intersect at the same point.



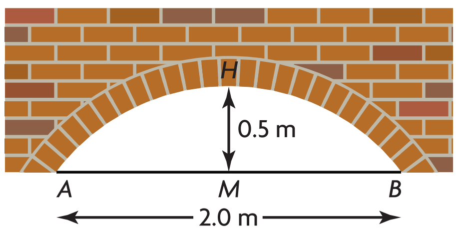
FURTHER *Your Understanding*

1. $\triangle ABC$ has vertices at $A(5, 1)$, $B(-2, 0)$, and $C(4, 8)$. Determine the coordinates of the point that is the same distance from each vertex.

2. Show how you know that a median divides a triangle into two smaller triangles that have the same area.

3. a) AB and CD have endpoints at $A(2, 9)$, $B(7, -6)$, $C(7, 6)$, and $D(2, -9)$, and they intersect at $E(5, 0)$. Use the products of the lengths of line segments to show that AB and CD are chords in the same circle.

   b) A bricklayer is constructing a circular arch with the dimensions shown. Use intersecting chords to determine the radius of the circle he will use.



4. Some similar figures that are constructed on the sides of a right triangle have an area relationship like the Pythagorean theorem.

   a) Construct a right triangle with perpendicular sides that are 6 cm and 8 cm and a hypotenuse that is 10 cm.

   b) Construct each figure in the table shown on the next page on all three sides of the right triangle. Calculate the areas of the similar figures. Copy and complete the table.

Similar Figures	Diagram	A_1 = Area on 6 cm Side (cm^2)	A_2 = Area on 8 cm Side (cm^2)	$A_1 + A_2$ (cm^2)	Area on Hypotenuse (cm^2)
square					
semicircle					
rectangle					
equilateral triangle					
right triangle					
parallelogram					

c) Change the dimensions of the right triangle to investigate whether this has any effect on the area relationships for each figure.

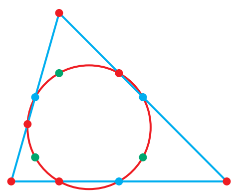
d) Explain what you have discovered.

YOU WILL NEED

- grid paper
- ruler

Curious Math

The Nine-Point Circle



A circle that passes through nine different points can be constructed in every triangle. These points can always be determined using the same strategy.

1. On a grid, draw a triangle with vertices at $P(1, 11)$, $Q(9, 8)$, and $R(1, 2)$. Determine the midpoints of the sides, and mark each midpoint with a blue dot.
2. Determine the equation of each altitude. Then determine the coordinates of the point where the altitude meets the side that is opposite each vertex. Mark these points on your diagram in red.
3. Determine the coordinates of the orthocentre (the point where all three altitudes intersect). Then, for each altitude, determine the coordinates of the midpoint of the line segment that joins the orthocentre to the vertex. Mark these midpoints on your diagram in green.
4. Determine the coordinates of the circumcentre (the point where all three perpendicular bisectors intersect). Determine the midpoint of the line segment that joins the circumcentre to the orthocentre. Mark this midpoint with the letter N. This is the centre of your nine-point circle. Draw the circle.
5. Identify how each of the points that lie on the nine-point circle can be determined for any triangle.

Similar Figures	Diagram	A_1 = Area on 6 cm Side (cm^2)	A_2 = Area on 8 cm Side (cm^2)	$A_1 + A_2$ (cm^2)	Area on Hypotenuse (cm^2)
square					
semicircle					
rectangle					
equilateral triangle					
right triangle					
parallelogram					

c) Change the dimensions of the right triangle to investigate whether this has any effect on the area relationships for each figure.

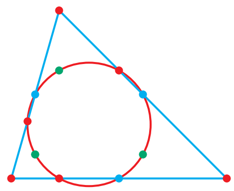
d) Explain what you have discovered.

YOU WILL NEED

- grid paper
- ruler

Curious | Math

The Nine-Point Circle



A circle that passes through nine different points can be constructed in every triangle. These points can always be determined using the same strategy.

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5. Identify how each of the points that lie on the nine-point circle can be determined for any triangle.

2.7 Using Coordinates to Solve Problems

GOAL

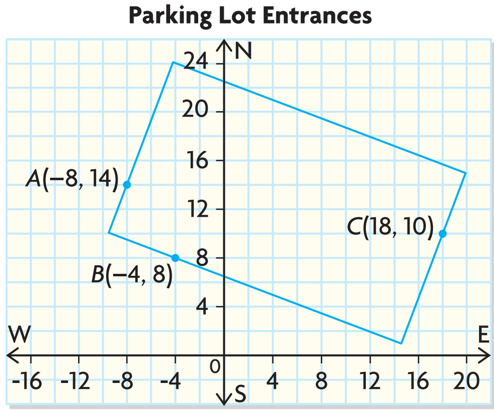
Use properties of lines and line segments to solve problems.

YOU WILL NEED

- grid paper
- ruler

LEARN ABOUT the Math

Rebecca is designing a parking lot. A tall mast light will illuminate the three entrances, which will be located at points A, B, and C. Rebecca needs to position the lamp so that it illuminates each entrance equally.



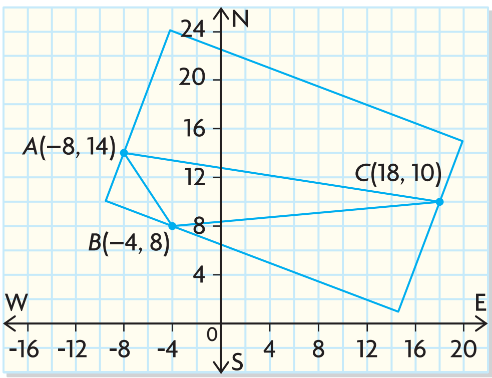
? How can Rebecca determine the location of the lamp?

EXAMPLE 1 **Solving a problem** using a triangle property

Determine the location of the lamp in the parking lot that Rebecca is designing.

Jack's Solution

The lamp should be placed the same distance from all three vertices of $\triangle ABC$.



If the lamp is the same distance from all three vertices, I reasoned that it would be at the centre of a circle that passes through all three vertices. I remembered that this point occurs where the perpendicular bisectors of the sides of the triangle intersect.

I decided to determine the perpendicular bisectors of AB and BC. I started with AB.

The midpoint of AB is

$$\left(\frac{-8 + (-4)}{2}, \frac{14 + 8}{2}\right) = (-6, 11).$$

To write an equation, I needed the slope and one point on the perpendicular bisector. I knew that the midpoint of AB would be on the perpendicular bisector, so I calculated this first.

$$m_{AB} = \frac{8 - 14}{-4 - (-8)}$$
$$= \frac{-6}{4}$$
$$= -\frac{3}{2}$$

The slope of the perpendicular bisector of AB is $\frac{2}{3}$.

Because the bisector is perpendicular to AB, its slope is the negative reciprocal of the slope of AB.

I calculated the slope of AB. Then I figured out the negative reciprocal to get the slope of the perpendicular bisector.

An equation is $y = \frac{2}{3}x + b$.

$$11 = \frac{2}{3}(-6) + b$$
$$11 = -4 + b$$
$$15 = b$$

The equation of the perpendicular bisector of AB is $y = \frac{2}{3}x + 15$.

I wrote an equation of a line with slope $\frac{2}{3}$, then I substituted the coordinates of the midpoint into the equation to determine the value of b.

The midpoint of BC is $\left(\frac{-4 + 18}{2}, \frac{8 + 10}{2}\right) = (7, 9)$.

I did the same calculations for BC.

$$m_{BC} = \frac{10 - 8}{18 - (-4)}$$
$$= \frac{2}{22}$$
$$= \frac{1}{11}$$

The slope of the perpendicular bisector of BC is -11.

The negative reciprocal of $\frac{1}{11}$ is -11.

An equation is $y = -11x + b$.

$$9 = -11(7) + b$$
$$9 = -77 + b$$
$$86 = b$$

The equation of the perpendicular bisector of BC is $y = -11x + 86$.

$$y = \frac{2}{3}x + 15$$

$$y = -11x + 86$$

At the point of intersection,

$$\frac{2}{3}x + 15 = -11x + 86$$

$$\frac{35}{3}x = 71$$

$$3\left(\frac{35}{3}\right)x = 3(71)$$

$$35x = 213$$

$$x = \frac{213}{35}$$

$$x \doteq 6.09$$

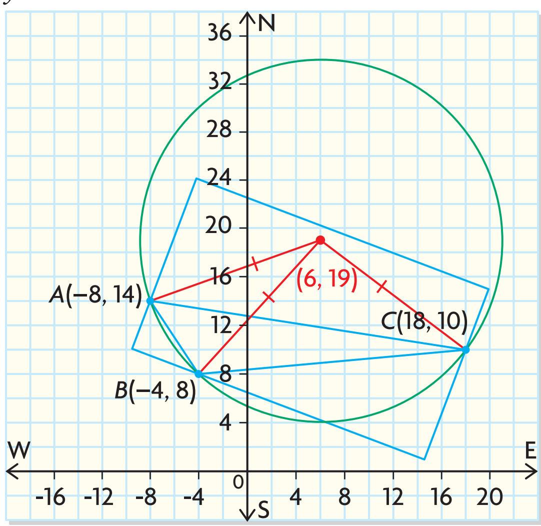
$$y = \frac{2}{3}\left(\frac{213}{35}\right) + 15$$

$$y = \frac{142}{35} + 15$$

$$y \doteq 19.06$$

To determine where the two perpendicular bisectors intersect, I set up their equations as a system of equations. I used the method of substitution to solve this system of equations.

First, I determined *x*. Then I substituted the value of *x* into the equation of the perpendicular bisector of *AB* to determine the value of *y*. I used the fractional value for *x* to minimize any rounding error.



If the lamp is placed at (6, 19), it will be about the same distance from each entrance. It will illuminate each entrance equally.

I rounded the values of *x* and *y* to the nearest integer.

Reflecting

A. Why is the intersection of two of the perpendicular bisectors the centre of the circle that Rebecca wants?

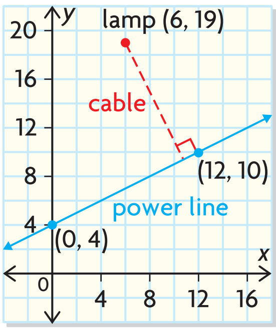
B. Why did Jack only need to determine the intersection of two of the perpendicular bisectors for the triangle?

APPLY the Math

EXAMPLE 2 Solving a problem using coordinates

The closest power line to the parking lot in Example 1 runs along a straight line that contains points (0, 4) and (12, 10). At what point on the power line should the cable from the lamp be connected? If each unit represents 1 m, how much cable will be needed to reach the power line? Round your answers to the nearest tenth.

Eden's Solution



I drew a diagram. The shortest distance from the lamp to the power line is the perpendicular distance. I drew this on my diagram.

To calculate the perpendicular distance, I had to determine the point where the perpendicular line intersects the power line. To do this, I had to determine the equations for the cable and the power line.

$$m = \frac{10 - 4}{12 - 0}$$

$$= \frac{6}{12}$$

$$= \frac{1}{2}$$

I determined the equation for the power line first. I already knew that the y-intercept is 4, so I just had to calculate the slope.

The equation of the power line is $y = \frac{1}{2}x + 4$.

The cable is perpendicular to the power line, so the slope of the equation for the power line is -2.

The cable from the lamp is perpendicular to the power line. The slope of the equation for the cable is the negative reciprocal of $\frac{1}{2}$, which is -2.

Therefore, an equation for the perpendicular line is $y = -2x + b$.

The point (6, 19) is on this line, so

$19 = -2(6) + b$

$19 = -12 + b$

$31 = b$

I used this slope to write an equation for the cable. Then I substituted the coordinates for the lamp into the equation to determine the value of b.

The equation of the perpendicular line from (6, 19) to the power line is $y = -2x + 31$.

$$y = \frac{1}{2}x + 4$$
$$y = -2x + 31$$
$$\frac{1}{2}x + 4 = -2x + 31$$
$$\frac{5}{2}x = 27$$
$$x = \frac{54}{5}$$
$$x = 10.8$$

I used substitution to solve the system of equations and determine the point where the two lines intersect.

The corresponding value of y is $y = \frac{1}{2}(10.8) + 4$
$= 9.4$

The cable from the lamp should be connected to the power line at point (10.8, 9.4).

$$\begin{aligned}\text{Length of cable} &= \sqrt{(10.8 - 6)^2 + (9.4 - 19)^2} \\ &= \sqrt{23.04 + 92.16} \\ &= \sqrt{115.2} \\ &\doteq 10.73\end{aligned}$$

I used the distance formula to calculate the length of cable that will be needed. I rounded my answer up to the nearest tenth of a metre to make sure I had extra cable.

About 10.8 m of cable will be needed to connect the lamp to the power line.

In Summary

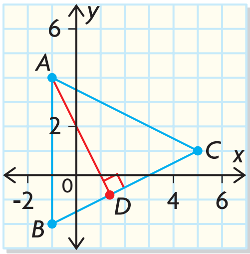
Key Idea

- You can use the properties of lines and line segments to solve multi-step problems when you can use coordinates for some or all of the given information in the problem.

Need to Know

- When solving a multi-step problem, you may find it helpful to follow these steps:
  - Read the problem carefully, and make sure that you understand it.
  - Make a plan to solve the problem, and record your plan.
  - Carry out your plan, and try to keep your work organized.
  - Look over your solution, and check that your answers seem reasonable.
- Drawing a graph and labelling it with the given information may help you plan your solution and check your results.
- You may need to determine the coordinates of a point of intersection before using the formulas for the slope and length of a line segment.

CHECK Your Understanding



Questions 1 to 5 refer to the diagram at the left.

$\triangle ABC$ has vertices at $A(-1, 4)$, $B(-1, -2)$, and $C(5, 1)$. The altitude from vertex A meets BC at point D.

1. **a)** Determine the slope of BC.
b) Determine the slope of AD.
c) Determine the equation of the line that contains AD.

2. Determine the equation of the line that contains BC.

3. Determine the coordinates of point D.

4. Determine the lengths of BC and AD.

5. Determine the area of $\triangle ABC$.

PRACTISING

6. A triangle has vertices at $A(-3, 2)$, $B(-5, -6)$, and $C(5, 0)$.
a) Determine the equation of the median from vertex A.
b) Determine the equation of the altitude from vertex A.
c) Determine the equation of the perpendicular bisector of BC.
d) What type of triangle is $\triangle ABC$? Explain how you know.

7. Points $P(-9, 2)$ and $Q(9, -2)$ are endpoints of a diameter of a circle.
K **a)** Write the equation of the circle.
b) Show that point $R(7, 6)$ is also on the circle.
c) Show that $\angle PRQ$ is a right angle.

8. $\triangle LMN$ has vertices at $L(3, 4)$, $M(4, -3)$, and $N(-4, -1)$. Use analytic geometry to determine the area of the triangle.

9. $\triangle DEF$ has vertices at $D(2, 8)$, $E(6, 2)$, and $F(-3, 2)$. Use analytic geometry to determine the coordinates of the orthocentre (the point where the altitudes intersect).

10. $\triangle PQR$ has vertices at $P(-12, 6)$, $Q(4, 0)$, and $R(-8, -6)$. Use analytic geometry to determine the coordinates of the centroid (the point where the medians intersect).

11. $\triangle JKL$ has vertices at $J(-2, 0)$, $K(2, 8)$, and $L(7, 3)$. Use analytic geometry to determine the coordinates of the circumcentre (the point where the perpendicular bisectors intersect).

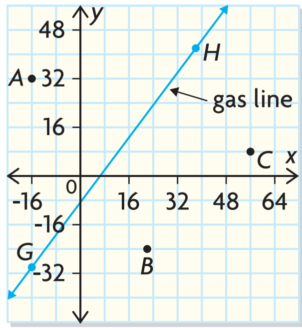
12. A university has three student residences, which are located at points $A(2, 2)$, $B(10, 6)$, and $C(4, 8)$ on a grid. The university wants to build a tennis court an equal distance from all three residences. Determine the coordinates of the tennis court.

13. Explain two different strategies you could use to show that points
C D, E, and F lie on the same circle, with centre C.

14. A design plan for a thin triangular computer component shows
A the vertices at points (8, 12), (12, 4), and (2, 8). Determine the coordinates of the centre of mass.

15. A stained glass window is in the shape of a triangle, with vertices at $A(-1, -2)$, $B(-2, 1)$, and $C(5, 0)$. $\triangle XYZ$ is formed inside $\triangle ABC$ by joining the midpoints of the three sides. The glass that is used for $\triangle XYZ$ is blue, but the remainder of $\triangle ABC$ is green. Determine the ratio of green to blue glass used.

16. Three homes in a rural area, labelled A, B, and C in the diagram at the right, are converting to natural gas heating. They will be connected to the gas line labelled GH in the diagram. On a plan marked out in metres, the coordinates of the points are $A(-16, 32)$, $B(22, -24)$, $C(56, 8)$, $G(-16, -30)$, and $H(38, 42)$.

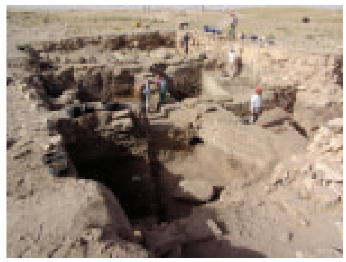


a) Determine the length of pipe that the gas company will need to connect the three houses to the gas line. Which homeowner will have the highest connection charge?

b) Determine the best location for a lamp to illuminate the three homes equally.

17. Determine the type of triangle that is formed by the lines $x + y = 11$, $x - y = 1$, and $x - 3y = 3$. Justify your decision.

18. Archaeologists on a dig have found an outside fragment of an ancient
T circular platter. They want to construct a replica of the platter for a display. How could they use coordinates to calculate the diameter of the platter? Include a diagram in your explanation.



19. Suppose that you know the coordinates of the vertices of a triangle. Describe the strategy you would use to determine the equation of each median and altitude that can be drawn from each vertex of the triangle to the opposite side.

Career *Connection*

An archaeologist searches for clues about the lives of people in past civilizations. Most archaeologists are employed by a university or a museum.

Extending

20. A triangle has vertices at $P(-1, 2)$, $Q(4, -4)$, and $R(1, 2)$. Show that the centroid divides each median in the ratio 2:1.

21. A circle is defined by the equation $x^2 + y^2 = 10a^2$.

a) Show that RQ, with endpoints $R(3a, a)$ and $Q(a, -3a)$, is a chord in the circle.

b) Show that the line segment joining the centre of the circle to the midpoint of RQ is perpendicular to RQ.

2 Chapter Review

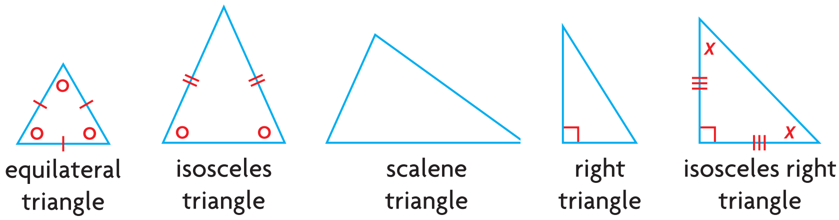
FREQUENTLY ASKED Questions

Study *Aid*

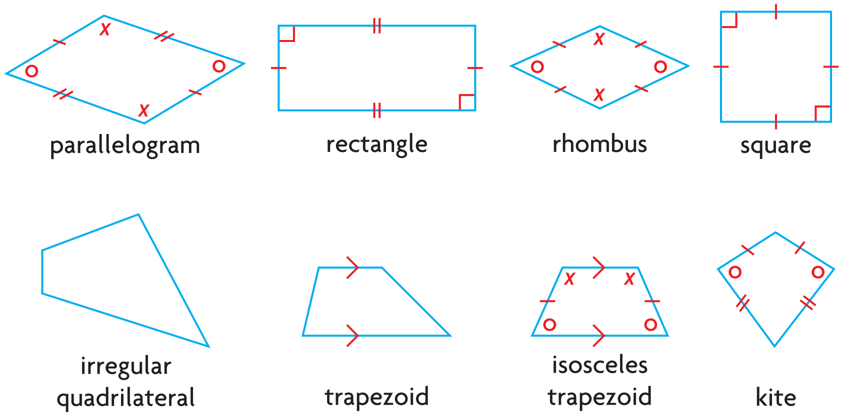
- See Lesson 2.4, Examples 1 and 2.
- Try Chapter Review Questions 12 to 15.

Q: How do you use the coordinates of the vertices of a triangle or quadrilateral to determine what type of figure it is?

A: To determine whether a triangle is isosceles, equilateral, or scalene, you calculate the side lengths using the distance formula. To determine whether the triangle is a right triangle, you substitute the side lengths into the Pythagorean theorem to see if they work, or you calculate the slopes of the line segments to see if two of the slopes are negative reciprocals.



For a quadrilateral, you determine the length and slope of each line segment that forms a side. Then you compare the lengths to see if there are equal sides, and compare the slopes to see if any sides are parallel or perpendicular.



Study *Aid*

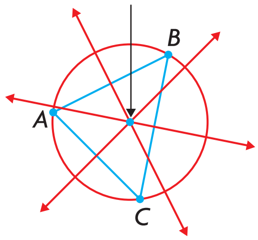
- See Lesson 2.5, Examples 1 to 3.
- Try Chapter Review Questions 16 to 20.

Q: How can you use the coordinates of vertices to verify properties of triangles, quadrilaterals, or circles?

A: You can use the coordinates of vertices to calculate midpoints and slopes, as well as side lengths in a triangle, lengths of sides or diagonals in a quadrilateral, or lengths of chords in a circle. Then you can use these values to verify properties of the figure.

Q: How do you use coordinates to locate a point that is the same distance from three given points?

A: You draw two line segments that join two pairs of given points. Then you determine the point of intersection of the perpendicular bisectors of these line segments. The point where the perpendicular bisectors intersect (called the circumcentre) is the same distance from the three given points.

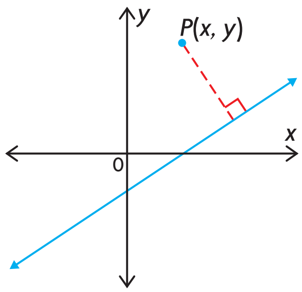


Study Aid

- See Lesson 2.7, Example 1.
- Try Chapter Review Question 23.

Q: How do you calculate the distance from a point to a line?

A: The distance from a point to a line is the perpendicular distance, since this is the shortest possible distance.



To calculate the distance from P to the line in the diagram:

- Determine the equation of a perpendicular line that goes through P. To do this, take the negative reciprocal of the slope of the line in the diagram. Then use the coordinates of P to determine the y-intercept of the perpendicular line.
- Determine the coordinates of the point of intersection of the line in the diagram and the perpendicular line by solving the linear system formed by the two lines.
- Determine the length of the line segment that joins P to this point of intersection using the distance formula.

Study Aid

- See Lesson 2.3, Example 3, and Lesson 2.7, Example 2.
- Try Chapter Review Question 25.

PRACTICE Questions

Lesson 2.1

1. On the design plan for a garden, a straight path runs from $(-25, 20)$ to $(40, 36)$. A lamp is going to be placed at the midpoint of the path. Determine the coordinates for the lamp.

2. $\triangle ABC$ has vertices at $A(-4, 4)$, $B(-4, -2)$, and $C(2, -2)$.
   a) Determine the equation of the median from B to AC.
   b) Is the median for part a) also an altitude? Explain how you know.

3. $\triangle LMN$ has vertices at $L(0, 4)$, $M(-5, 2)$, and $N(2, -2)$. Determine the equation of the perpendicular bisector that passes through MN.

Lesson 2.2

4. Which point is closer to the origin: $P(-24, 56)$ or $Q(35, -43)$?

5. A builder needs to connect a partially built house to a temporary power supply. On the plan, the coordinates of the house are (20, 110) and the coordinates of the power supply are (105, 82). What is the least amount of cable needed?

6. $\triangle QRS$ has vertices at $Q(2, 6)$, $R(-3, 1)$, and $S(6, 2)$. Determine the perimeter of the triangle.

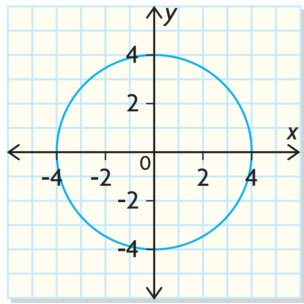
7. $\triangle XYZ$ has vertices at $X(1, 6)$, $Y(-3, 2)$, and $Z(9, 4)$. Determine the length of the longest median in the triangle.

Lesson 2.3

8. **a)** Determine the equation of the circle that is centred at (0, 0) and passes through point $(-8, 15)$.
   b) Identify the coordinates of the intercepts and three other points on the circle.

9. A circle has a diameter with endpoints $C(20, -21)$ and $D(-20, 21)$. Determine the equation of the circle.

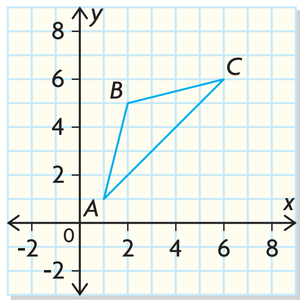
10. Determine the equation of this circle.



11. The point $(-2, k)$ lies on the circle $x^2 + y^2 = 20$. Determine the values of k. Show all the steps in your solution.

Lesson 2.4

12. $\triangle ABC$ has vertices as shown. Use analytic geometry to show that $\triangle ABC$ is isosceles.



13. A triangle has vertices at $A(1, 1)$, $B(-2, -1)$, and $C(3, -2)$. Calculate the side lengths to determine whether the triangle is isosceles, equilateral, or scalene.

14. Show that the quadrilateral with vertices at $J(-1, 1)$, $K(3, 4)$, $L(8, 4)$, and $M(4, 1)$ is a rhombus.

15. Determine the type of quadrilateral described by the vertices $R(-3, 2)$, $S(-1, 6)$, $T(3, 5)$, and $U(1, 1)$. Show all the steps in your solution.

Lesson 2.5

16. A quadrilateral has vertices at $A(-3, 1)$, $B(-5, -9)$, $C(7, -1)$, and $D(3, 3)$. Show that the midsegments of the quadrilateral form a parallelogram.

17. Show that points (10, 10), $(-7, 3)$, and $(0, -14)$ lie on a circle with centre $(5, -2)$.

18. A triangle has vertices at $P(-2, 7)$, $Q(-4, 2)$, and $R(6, -2)$.

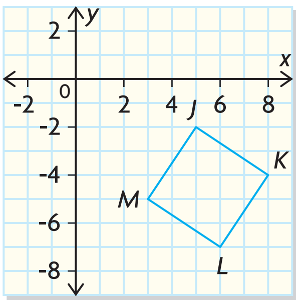
a) Show that $\triangle PQR$ is a right triangle.

b) Show that the midpoint of the hypotenuse is the same distance from each vertex.

19. a) Show that points (6, 7) and $(-9, 2)$ are the endpoints of a chord in a circle with centre (0, 0).

b) A line is drawn through the centre of the circle so that it is perpendicular to the chord. Verify that this line passes through the midpoint of the chord.

20. a) Quadrilateral *JKLM* has vertices as shown. Show that the diagonals of the quadrilateral bisect each other.

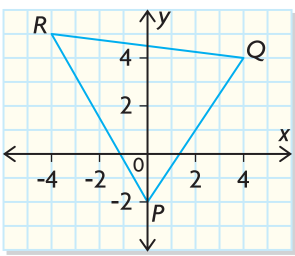


b) Make a conjecture about the type of quadrilateral *JKLM* could be.

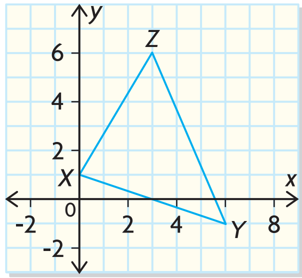
c) Use analytic geometry to verify your conjecture.

Lesson 2.7

21. $\triangle PQR$ has vertices at $P(0, -2)$, $Q(4, 4)$, and $R(-4, 5)$. Use analytic geometry to determine the coordinates of the orthocentre (the point where the altitudes intersect).



22. $\triangle XYZ$ has vertices at $X(0, 1)$, $Y(6, -1)$, and $Z(3, 6)$. Use analytic geometry to determine the coordinates of the centroid (the point where the medians intersect).



23. A new lookout tower is going to be built so that it is the same distance from three ranger stations. If the stations are at $A(-90, 28)$, $B(0, -35)$, and $C(125, 20)$ on a grid, determine the coordinates of the point where the new tower should be built.

24. Predict the type of quadrilateral that is formed by the points of intersection of the lines $3x + y - 4 = 0$, $4x - 5y + 30 = 0$, $y = -3x - 1$, and $-4x + 5y + 10 = 0$. Give reasons for your prediction. Verify that your prediction is correct by solving this problem.

25. A builder wants to run a temporary line from the main power line to a point near his site office. On the site plan, the site office is at $S(25, 18)$ and the main power line goes through points $T(1, 5)$ and $U(29, 12)$. Each unit represents 1 m.

a) At what point should the builder connect to the main power line?

b) What length of cable will the builder need?

2 Chapter Self-Test

Process Checklist

- ✔ Questions 1, 2, 6, and 8: Did you make **connections** between analytical geometry and the situation?
- ✔ Questions 3 and 4: Did you apply **reasoning** skills to construct a mathematical argument to confirm each figure type?
- ✔ Question 7: Did you use appropriate mathematical vocabulary to **communicate** your thinking?

1. An underground cable is going to be laid between points $A(-6, 23)$ and $B(14, -12)$.

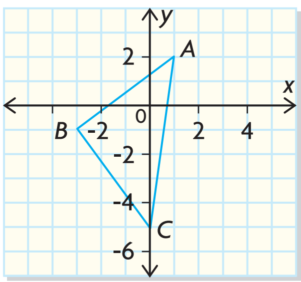
a) If each unit represents 1 m, what length of cable will be needed? Give your answer to the nearest metre.

b) An access point will be located halfway between the endpoints of the cable. At what coordinates should the access point be built?

2. A stone is tossed into a pond, creating a circular ripple. The radius of the ripple increases by 12 cm/s.

a) Write an equation that describes the ripple exactly 3 s after the stone lands in the water. Use the origin as the point where the stone lands in the water.

b) A bulrush is located at point $(-36, 48)$. When will the ripple reach the bulrush?



3. The triangle at the left has vertices at $A(1, 2)$, $B(-3, -1)$, and $C(0, -5)$. Use analytic geometry to show that the triangle is an isosceles right triangle.

4. The corners of a building lot are marked at $P(-39, 39)$, $Q(-78, -13)$, $R(26, -91)$, and $S(65, -39)$ on a grid.

a) Verify that *PQRS* is a rectangle.

b) What is the perimeter of the building?

5. Quadrilateral *JKLM* has vertices at $J(2, 4)$, $K(6, 1)$, $L(2, -2)$, and $M(-2, 1)$. What type of quadrilateral is *JKLM*?

6. Three straight paths in a park form a triangle with vertices at $A(-24, 16)$, $B(56, -16)$, and $C(-72, -32)$. A new fountain is the same distance from the intersections of the three paths. Determine the location of the new fountain.

7. Explain how you can use analytic geometry to calculate the distance from a known point to a line that passes through two other known points.

8. The sides of a triangle are defined by the equations $x + 2y - 2 = 0$, $2x - y - 4 = 0$, and $3x + y + 9 = 0$. Determine the type of triangle that is formed by these three sides.

X Marks the Spot

A new diagnostic centre, with laboratories and computer-imaging equipment, is being planned. The centre will serve four walk-in clinics.

On a map, these clinics are located at $A(1, 12)$, $B(12, 19)$, $C(19, 8)$, and $D(3, 2)$. On the map, 1 unit represents 1 km.

? Where should the diagnostic centre be located?

A. On a grid, show the locations of the walk-in clinics. Estimate where you think the diagnostic centre should go, and mark this point.

B. Use coordinates to determine a point that is the same distance from the clinics at points A, B, and C.

C. Calculate the distance from the clinic at point D to the point you determined for part B.

D. Use coordinates to determine a point that is the same distance from the clinics at points A, B, and D.

E. Calculate the distance from the clinic at point C to the point you determined for part D.

F. Repeat parts D and E for the other combinations of three clinics.

G. Based on your results, choose the best location for the diagnostic centre. Justify your choice.

Safety *Connection*

A protective cabinet, as well as protective gloves and jacket, are needed for handling materials in a laboratory.

Task *Checklist*

- ✔ Did you show all your steps?
- ✔ Did you draw and label your diagram accurately?
- ✔ Did you support your choice of location for the diagnostic centre?
- ✔ Did you explain your thinking clearly?